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Notebooks of
Srinivasa Ramanujan

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विषय संख्या.....आगत नं०.....
लेखक.....
शीर्षक Note book of Srinivasa Ramanujan

दिनांक	सदस्य संख्या	दिनांक	सदस्य संख्या

गुरुकुल कांगड़ी विश्वविद्यालय
कृपया पुस्तक के ऊपर कोई निशान आदि न लगायें।

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VOLUME I

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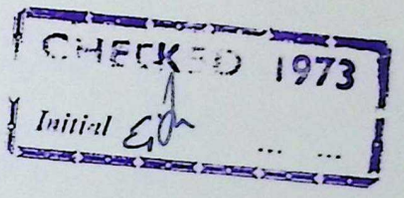

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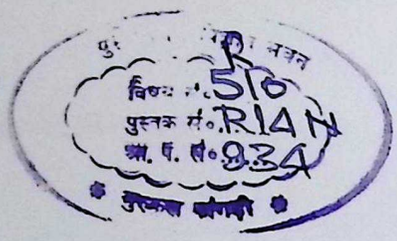

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PREFACE

The publication of this facsimile edition of the manuscripts of the late Srinivasa Ramanujan, in two volumes, has been made possible by the co-operative effort of three institutions : the University of Madras, the Tata Institute of Fundamental Research, and the Sir Dorabji Tata Trust. The suggestion to bring out this publication was made by Professor K. Chandrasekharan to Dr. H. J. Bhabha, Director of the Tata Institute of Fundamental Research, who got in touch with Dr. A. L. Mudaliar, Vice-Chancellor of the University of Madras, and Professor R. Choksi, Managing Trustee of the Sir Dorabji Tata Trust. The manuscripts were readily made available by the University, and the funds by the Trust. Professor Chandrasekharan has been in charge of the arrangements for the compilation, production, and distribution of the edition. Mr. S. Ramu and Mr. R. Subbu of the Commercial Printing Press have done the difficult work of reproducing the manuscripts.

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NOTEBOOK 1

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CHAPTER I.

MAGIC SQUARES.

Let a be the average, S a row or a column, m middle row or rows or column or columns, d a diagonal, and W the whole sum.

I. When the Square contains 3 rows and 3 columns,

i. If S and d are equal, write a in the middle and supply the other figures.

Sol:— $d_1 + d_2 + m_1 + m_2 = W + 3x$ where x is the req. figure in the middle.

$\therefore 4S = 3S + 3x. \quad \therefore S = 3x$ or $x = a$.

Cor. The figures in d are in A.P.

Sol:— The sum of the numbers in d is S or $3a$ and 2nd $= a$. $\therefore$ 1st + 3rd $= 2a =$ twice the second.

$\therefore$ they are in A.P. Similarly in m also.

Ex. 1. Fill up the Square when $S = 15$

6	1	8
7	5	3
2	9	4

2. When $S = 27$ and all numbers are odd.

15	1	11
5	9	13
7	17	3

ii. When s and d are unequal, Write $\frac{d_1 + d_2 - S}{3}$ in the middle

Ex. 1. Shew that the numbers in m are in A.P. here also

Sol. Proceed as in I. 1. i Cor.

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2. Fill up the Square when $S=20$, $d_1=16$ and $d_2=19$.

10	2	8
4	5	11
6	13	1

iii. When the diagonals, columns and rows are all different, write $\frac{1}{3}(d_1+d_2+m_1+m_2-W)$ in the middle.

Sol. As in I 1. i.

Ex. Fill up the Square when $d_1=15$, $d_2=19$ and the columns and rows are 16, 17, 12, 6, 21, and 18.

4	2	3
8	9	4
7	6	5

2. When an oblong contains 3 rows and 4 columns;

$A+C=2B+3D$			
A	C+D	A+2D	C+3D
B+6D	B+4D	B+2D	B
C	A+D	C+2D	A+3D

Ex: Fill up the oblong when $a=8$

7	13	3	15
11	9	7	5
12	2	14	4

3. When a Square contains 4 rows and 4 columns.

i. When the diagonals, columns and rows are all different, arrange the central four so that the sum may be $\frac{1}{2}(d_1+d_2+m_1+m_2-W)$.

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ii. When $s = d$,

Fig I

A+P	C+S	D+Q	B+R
D+R	B+Q	A+S	C+P
B+S	D+P	C+R	A+Q
C+Q	A+R	B+P	D+S

Fig II

A+D = B+C		P+S = Q+R	
A+P	D+Q	D+R	A+S
B+S	C+R	C+Q	B+P
C+S	B+R	B+Q	C+P
D+P	A+Q	A+R	B+S

Ex.

1	10	15	8
16	7	2	9
6	13	12	3
11	4	5	14

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

4. When a square contains 5 rows and 5 columns.

A+P	E+R	D+T	C+Q	B+S
C+T	B+Q	A+S	E+P	D+R
E+S	D+P	C+R	B+T	A+Q
B+R	A+T	E+Q	D+S	C+P
D+Q	C+S	B+P	A+R	E+T

1	58	59	4	5	62	63	8
16	55	54	13	12	57	50	9
24	47	46	21	20	43	42	17
25	34	35	28	29	38	39	32
33	26	27	36	37	30	31	40
48	23	22	45	44	19	18	41
56	15	14	53	52	11	10	49
57	2	3	60	61	6	7	64

Similarly we can form Squares Containing more rows & Columns.

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CHAPTER II

1. $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots\ldots + \frac{1}{2n}$

$= \frac{n}{2n+1} + \frac{1}{2^3-2} + \frac{1}{4^3-4} + \frac{1}{6^3-6} + \ldots\ldots + \frac{1}{(2n)^3-2n}$.

Sol:- $\frac{1}{(2n)^3-2n} = \frac{1}{2}\cdot\frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2(2n+1)}$.

$\therefore$ R.H.S $= \frac{1}{2}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right) - \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots\frac{1}{2n}\right)$

$+ \frac{1}{2}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n+1}\right) + \frac{n}{2n+1} - \frac{1}{2}$.

$= \left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right) - \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right)$.

$= \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots+\frac{1}{2n}\right) - \left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}\right)$

$= \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots\ldots + \frac{1}{2n}$.

Cor. $2\log_e 2 = 1 + \frac{2}{2^3-2} + \frac{2}{4^3-4} + \frac{2}{6^3-6} + \frac{2}{8^3-8} + \ldots$ ad inf.

Sol. R.H.S $= 2\left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{2n}\right)$ when $n=\infty$

Let $n = \frac{1}{dx}$.

then the given Series $= \frac{2dx}{1+dx} + \frac{2dx}{1+2dx} + \ldots\ldots + \frac{2dx}{1+1}$

$= 2\int_1^2 \frac{1}{x}\,dx = 2\log_e 2$.

or thus —

In the Solution of II 8. we get $\left(1+\frac{1}{3}+\frac{1}{5}+\ldots\frac{1}{2n-1}\right)$

$-\left(\frac{1}{2}+\frac{1}{4}+\ldots\ldots\frac{1}{2n}\right)$. When $n=\infty$ this becomes

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-$ &c $= \log_e 2$. $\therefore$ The req^d Sum $= 2\log_e 2$.

N.B. $\Sigma\frac{1}{n}$ means the sum of the reciprocals of n natural numbers. Therefore $\Sigma\frac{1}{an} = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{an}$ and $\neq \frac{1}{a}+\frac{1}{2a}+\frac{1}{3a}+\ldots+\frac{1}{na}$. $\Sigma\frac{1}{m/n}$ should not be written as $\Sigma\frac{n}{m}$ which has no meaning according to our Convention.

Ex. Shew that $\frac{n-1}{n+1} + \frac{n-2}{n+2} + \frac{n-3}{n+3} + \ldots\ldots + \frac{n-n}{n+n}$

$= 2n\left\{\frac{1}{1\cdot2\cdot3} + \frac{1}{3\cdot4\cdot5} + \frac{1}{5\cdot6\cdot7} + \ldots\ldots + \frac{1}{(2n-1)(2n)(2n+1)}\right\} - \frac{n}{2n+1}$

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Sol. We have by II/.

$\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} = \frac{n}{2n+1} + \frac{1}{1.2.3} + \frac{1}{3.4.5} + \dots\dots$

$+ \frac{1}{(2n-1)(2n)(2n+1)}$. Multiplying both sides by $2n$

$\frac{2n}{n+1} + \frac{2n}{n+2} + \dots + \frac{2n}{2n} = \frac{2n^2}{2n+1} + 2n\left\{\frac{1}{1.2.3} + \frac{1}{3.4.5} + \text{&c to } n \text{ terms}\right\}$.

$\therefore \left(\frac{2n}{n+1} - 1\right) + \left(\frac{2n}{n+2} - 1\right) + \left(\frac{2n}{n+3} - 1\right) + \dots + \left(\frac{2n}{2n} - 1\right)$

$= \frac{2n^2}{2n+1} - n + 2n\left\{\frac{1}{1.2.3} + \frac{1}{3.4.5} + \text{&c to } n \text{ terms}\right\}$.

$\therefore \frac{n-1}{n+1} + \frac{n-2}{n+2} + \frac{n-3}{n+3} + \dots + \frac{n-n}{n+n}$

$= 2n\left\{\frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{5.6.7} + \dots + \frac{1}{(2n-1)2n(2n+1)}\right\} - \frac{n}{2n+1}$.

2. $\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{3n+1} = 1 + \frac{2}{3^3-3} + \frac{2}{6^3-6} + \dots + \frac{2}{(3n)^3-3n}$.

Sol. By proceeding as in II/ we have R.H.S $= \Sigma\frac{1}{3n+1} - \Sigma\frac{1}{n} =$ L.H.S

Cor. $1 + \frac{2}{3^3-3} + \frac{2}{6^3-6} + \frac{2}{9^3-9} + \text{&c} = \log_e 3$.

3. $\tan^{-1}\frac{1}{n+1} + \tan^{-1}\frac{1}{n+2} + \dots + \tan^{-1}\frac{1}{3n+1}$

$= \tan^{-1}1 + \tan^{-1}\frac{10}{5.8} + \tan^{-1}\frac{20}{14.35} + \dots\dots + \tan^{-1}\frac{10n}{(3n^2+2)(9n^2}$

Cor. $\log_e 3 = \tan^{-1}1 + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{49} + \tan^{-1}\frac{3}{232} + \tan^{-1}\frac{4}{715} + \text{&}$

4. $\left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \dots \frac{1}{4n+1}\right)$

$= 1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \frac{2}{12^3-12} + \dots + \frac{2}{(4n)^3-4n}$

$= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{4n+1}\right) + \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\dots - \frac{1}{2n}\right)$.

Sol. By proceeding as in III/ we have R.H.S $= \Sigma\frac{1}{4n+1} - \frac{1}{2}\Sigma\frac{1}{2n}$

$- \frac{1}{2}\Sigma\frac{1}{n} = \Sigma\frac{1}{4n+1} - \Sigma\frac{1}{n} - \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}\right)$

$= \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{4n+1}\right) - \left(\frac{1}{2n+2} + \frac{1}{2n+4} + \dots + \frac{1}{4n}\right)$

$= \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \dots + \frac{1}{4n+1}\right)$

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Again $\Sigma\frac{1}{4n+1} - \frac{1}{2}\Sigma\frac{1}{2n} - \frac{1}{2}\Sigma\frac{1}{n} = \Sigma\frac{1}{4n+1} - \Sigma\frac{1}{2n} + \frac{1}{2}\Sigma\frac{1}{2n}$

$- \frac{1}{2}\Sigma\frac{1}{n} = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{4n+1}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{4n}\right)$

$+ \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n}\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right)$

$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{4n+1}$

$+ \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n}\right).$

Cor. $1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \frac{2}{12^3-12} + \&c = \frac{3}{2}\log_e 2.$

5. $\frac{2}{3}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{6n+1}\right)$

$= 1 + \frac{2}{6^3-6} + \frac{2}{12^3-12} + \frac{2}{18^3-18} + \cdots + \frac{2}{(6n)^3-6n}.$

Sol. Proceeding as in III the Sum is $\Sigma\frac{1}{6n+1} - \frac{1}{2}\Sigma\frac{1}{3n}$

$- \frac{1}{3}\Sigma\frac{1}{2n} - \frac{1}{6}\Sigma\frac{1}{n} = L.H.S.$

Cor. $1 + \frac{2}{6^3-6} + \frac{2}{12^3-12} + \frac{2}{18^3-18} + \&c = \frac{1}{2}\log_e 3 + \frac{1}{3}\log_e 4.$

Ex. 1. $\frac{1}{4}\log_e 2 = \frac{1}{2^3-2} + \frac{1}{6^2-6} + \frac{1}{10^3-10} + \frac{1}{14^3-14} + \&c$

2. $\log_e 2 = 1 - \frac{2}{2^3-2} + \frac{2}{4^3-4} - \frac{2}{6^3-6} + \&c$

3. $2\left\{1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \frac{2}{12^3-12} + \cdots + \frac{2}{(4n)^3-4n}\right\}$

$= \left\{1 + \frac{2}{2^3-2} + \frac{2}{4^3-4} + \cdots + \frac{2}{(4n)^3-4n}\right\} + \frac{1}{(4n+1)(4n+2)}$

$+ \frac{1}{2}\left\{1 + \frac{2}{2^3-2} + \frac{2}{4^3-4} + \frac{2}{6^3-6} + \cdots + \frac{2}{(2n)^3-2n}\right\}.$

4. Shew that $1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \cdots + \frac{2}{(4n)^3-4n}$

$= \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+2} + \cdots + \frac{1}{4n+1}\right)$

5. $\frac{1}{3^3-3} + \frac{1}{9^3-9} + \frac{1}{15^3-15} + \&c = \frac{1}{4}\log_e 3 - \frac{1}{3}\log_e 2.$

6. $\frac{4}{3}\log_e 2 = 1 - \frac{2}{3^3-3} + \frac{2}{6^3-6} - \frac{2}{9^3-9} + \&c.$

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7. Shew that $2\left\{1+\frac{2}{6^3-6}+\frac{2}{12^3-12}+\cdots+\frac{2}{(6n)^3-6n}\right\}$ 7

$+\frac{1}{3}\left\{1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\frac{2}{6^3-6}+\cdots+\frac{2}{(2n)^3-2n}\right\}$

$=1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\cdots+\frac{2}{(3n)^3-3n}+\frac{2}{(6n+1)(6n+2)(6n+3)}$

$+1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\cdots+\frac{2}{(6n)^3-6n}$.

8. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{7}$

$+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{9}+\tan^{-1}\frac{1}{10}+\tan^{-1}\frac{1}{11}+\tan^{-1}\frac{1}{12}+\tan^{-1}\frac{1}{13}$

$=\frac{\pi}{2}+2\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{49}+\tan^{-1}\frac{3}{232}+\tan^{-1}\frac{4}{715}$.

9. $2\left(\tan^{-1}\frac{1}{n+1}+\tan^{-1}\frac{1}{n+2}+\cdots\tan^{-1}\frac{1}{2n+1}\right)=\tan^{-1}\frac{n+1}{n}$

$+\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{4}{137}+\tan^{-1}\frac{6}{667}+\tan^{-1}\frac{8}{2081}+$

$\cdots+\tan^{-1}\frac{2n}{8n^4+2n^2+1}+$

$2\left(\tan^{-1}\frac{1}{1.7}+\tan^{-1}\frac{1}{2.19}+\tan^{-1}\frac{1}{3.39}+\cdots+\tan^{-1}\frac{1}{n(4n^2+3)}\right)$

10. $\tan^{-1}\frac{1}{n+1}+\tan^{-1}\frac{1}{n+2}+\cdots+\tan^{-1}\frac{1}{2n+1}+\tan^{-1}\frac{1}{2n+3}$

$+\tan^{-1}\frac{1}{2n+5}+\cdots+\tan^{-1}\frac{1}{4n+1}$

$=\frac{\pi}{4}+\tan^{-1}\frac{9}{53}+\tan^{-1}\frac{18}{599}+\tan^{-1}\frac{27}{2789}+\cdots+\tan^{-1}\frac{9n}{3(2n^2+2n)-1}$

$+\tan^{-1}\frac{4}{137}+\tan^{-1}\frac{8}{2081}+\tan^{-1}\frac{12}{10441}+\cdots\tan^{-1}\frac{4n}{128n^4+8n^2+1}$

N.B. $1+\frac{2}{x^3-x}+\frac{2}{(2x)^3-2x}+\frac{2}{(3x)^3-3x}+\cdots+\frac{2}{(nx)^3-nx}$ cannot be expressed as in II 2. for all values of x but 2, 3, 4 and 6 though it can be summed up for all values of x when $n=\infty$. Refer to chapter

6. If $K_n=3^n(n+\frac{1}{2})-\frac{1}{2}$, then

$\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{K_n}=$

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$$n\left\{1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\frac{2}{9^3-9}+\cdots+\frac{2}{(3n)^3-3n}\right\}$$

$$+(n-1)\left\{\frac{2}{(3K_0+3)^3-(3K_0+3)}+\frac{2}{(3K_0+6)^3-(3K_0+6)}+\cdots+\frac{2}{(3K_1)^3-3K_1}\right\}$$

$$+(n-2)\left\{\frac{2}{(3K_1+3)^3-(3K_1+3)}+\frac{2}{(3K_1+6)^3-(3K_1+6)}+\cdots+\frac{2}{(3K_2)^3-3K_2}\right\}$$

$+\cdots$ to n terms.

Sol. By II 2 we have,

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}=1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\cdots+\frac{2}{(3n)^3-3n}$$

$$\frac{1}{3n+2}+\frac{1}{3n+3}+\cdots+\frac{1}{9n+4}=1+\frac{2}{3^3-3}+\cdots+\frac{2}{(9n+3)^3-(9n+3)}$$

$$\frac{1}{9n+5}+\frac{1}{9n+6}+\cdots+\frac{1}{27n+13}=1+\frac{2}{8^3-3}+\cdots+\frac{2}{(27n+12)^3-(27n+12)}$$

Writing thus n times and then adding up all the terms we can get the result.

Cor. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{\frac{3^n-1}{2}}$

$$=n+(n-1)\left(\frac{2}{3^3-3}\right)+(n-2)\left(\frac{2}{6^3-6}+\frac{2}{9^3-9}+\frac{2}{12^3-12}\right)$$

$$+(n-3)\left(\frac{2}{15^3-15}+\frac{2}{18^3-18}+\cdots+\frac{2}{39^3-39}\right)+\text{&c to } n \text{ terms}.$$

N. B. II 6 & II 6 Cor. are very useful in finding the approximate value of $\Sigma\frac{1}{n}$ whether n is small or very great without knowing logarithms, differential and integral Calculus. In finding $\Sigma\frac{1}{n}$ it must be remembered that when a_1 and a_n are very great and a_1, a_2, a_3 &c are in A.P. the approximate value of $\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}$

$$=\frac{2n}{a_1+a_n}.$$

Ex. 1. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}$

$$=3+\frac{1}{6}+\frac{1}{105}+\frac{1}{360}+\frac{1}{858}.$$

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2. Shew that $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{1000} = 7\frac{1}{2}$ very nearly.

7. $\tan^{-1}\frac{2}{(n+1)^2}+\tan^{-1}\frac{2}{(n+3)^2}+\tan^{-1}\frac{2}{(n+5)^2}+$ &c to n terms $= \tan^{-1}\frac{2n}{n^2+2n+1}$.

Sol. $\tan^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{n+2}=\tan^{-1}\frac{2}{(n+1)^2}$

$\therefore$ L.H.S $= \left(\tan^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{n+2}\right)+\left(\tan^{-1}\frac{1}{n+2}-\tan^{-1}\frac{1}{n+4}\right)$

$+\left(\tan^{-1}\frac{1}{n+4}-\tan^{-1}\frac{1}{n+6}\right)+\ldots+\left(\tan^{-1}\frac{1}{n+2n-2}-\tan^{-1}\frac{1}{n+2n}\right)$

$= \tan^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{n+2n}=\tan^{-1}\frac{2n}{n^2+2n+1}$.

Cor. $\tan^{-1}\frac{2}{(n+1)^2}+\tan^{-1}\frac{2}{(n+3)^2}+\tan^{-1}\frac{2}{(n+5)^2}+$ &c $=\tan^{-1}\frac{1}{n}$.

Sol. Make n infinite in II 7.

Ex. 1. $\tan^{-1}\frac{2}{(n+1)^2}+\tan^{-1}\frac{2}{(n+2)^2}+\tan^{-1}\frac{2}{(n+3)^2}+$ &c $=\tan^{-1}\frac{2n+1}{n^2+n}$

Sol. $\tan^{-1}\frac{2}{(n+1)^2}+\tan^{-1}\frac{2}{(n+3)^2}+$ &c $=\tan^{-1}\frac{1}{n}$

$\therefore \tan^{-1}\frac{2}{(n+2)^2}+\tan^{-1}\frac{2}{(n+4)^2}+$ &c $=\tan^{-1}\frac{1}{n+1}$

$\therefore \tan^{-1}\frac{2}{(n+1)^2}+\tan^{-1}\frac{2}{(n+2)^2}+$ &c $=\tan^{-1}\frac{2n+1}{n^2+n-1}$.

N.B. If $n < \frac{\sqrt{5}-1}{2}$ add π to R.H.S.

2. $\tan^{-1}\frac{2}{(n+1)^2}-\tan^{-1}\frac{2}{(n+2)^2}+\tan^{-1}\frac{2}{(n+3)^2}-$ &c $=\tan^{-1}\frac{1}{n^2+n+1}$

3. $\tan^{-1}\frac{1}{2(n+1)^2}+\tan^{-1}\frac{1}{2(n+2)^2}+\tan^{-1}\frac{1}{2(n+3)^2}+$ &c $=\tan^{-1}\frac{1}{2n+1}$

4. $\frac{3\pi}{4}=\tan^{-1}\frac{2}{1^2}+\tan^{-1}\frac{2}{2^2}+\tan^{-1}\frac{2}{3^2}+$ &c

5. $\frac{\pi}{4}=\tan^{-1}\frac{1}{2\cdot1^2}+\tan^{-1}\frac{1}{2\cdot2^2}+$ &c $=\tan^{-1}\frac{2}{1^2}-\tan^{-1}\frac{2}{2^2}+$ &c

6. $\frac{\pi}{8}=\tan^{-1}\frac{1}{(1+\sqrt{2})^2}+\tan^{-1}\frac{1}{(1+2\sqrt{2})^2}+\tan^{-1}\frac{1}{(1+3\sqrt{2})^2}+$ &c

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2. Shew that $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{1000} = 7\frac{1}{2}$ very nearly.

7. $\tan^{-1}\frac{2}{(n+1)^2}+\tan^{-1}\frac{2}{(n+3)^2}+\tan^{-1}\frac{2}{(n+5)^2}+$ &c to n terms $= \tan^{-1}\frac{2n}{n^2+2n+1}$.

Sol. $\tan^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{n+2}=\tan^{-1}\frac{2}{(n+1)^2}$

$\therefore$ L.H.S $= \left(\tan^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{n+2}\right)+\left(\tan^{-1}\frac{1}{n+2}-\tan^{-1}\frac{1}{n+4}\right)$

$+\left(\tan^{-1}\frac{1}{n+4}-\tan^{-1}\frac{1}{n+6}\right)+\ldots+\left(\tan^{-1}\frac{1}{n+2n-2}-\tan^{-1}\frac{1}{n+2n}\right)$

$= \tan^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{n+2n}=\tan^{-1}\frac{2n}{n^2+2n+1}$.

Cor. $\tan^{-1}\frac{2}{(n+1)^2}+\tan^{-1}\frac{2}{(n+3)^2}+\tan^{-1}\frac{2}{(n+5)^2}+$ &c $=\tan^{-1}\frac{1}{n}$.

Sol. Make n infinite in II 7.

Ex. 1. $\tan^{-1}\frac{2}{(n+1)^2}+\tan^{-1}\frac{2}{(n+4)^2}+\tan^{-1}\frac{2}{(n+3)^2}+$ &c $=\tan^{-1}\frac{2n+1}{n^2+n}$

Sol. $\tan^{-1}\frac{2}{(n+1)^2}+\tan^{-1}\frac{2}{(n+3)^2}+$ &c $=\tan^{-1}\frac{1}{n}$

$\therefore \tan^{-1}\frac{2}{(n+2)^2}+\tan^{-1}\frac{2}{(n+4)^2}+$ &c $=\tan^{-1}\frac{1}{n+1}$

$\therefore \tan^{-1}\frac{2}{(n+1)^2}+\tan^{-1}\frac{2}{(n+2)^2}+$ &c $=\tan^{-1}\frac{2n+1}{n^2+n-1}$.

N.B. If $n < \frac{\sqrt{5}-1}{2}$ add π to R.H.S.

2. $\tan^{-1}\frac{2}{(n+1)^2}-\tan^{-1}\frac{2}{(n+2)^2}+\tan^{-1}\frac{2}{(n+3)^2}-$ &c $=\tan^{-1}\frac{1}{n^2+n+1}$

3. $\tan^{-1}\frac{1}{2(n+1)^2}+\tan^{-1}\frac{1}{2(n+2)^2}+\tan^{-1}\frac{1}{2(n+3)^2}+$ &c $=\tan^{-1}\frac{1}{2n+1}$

4. $\frac{3\pi}{4}=\tan^{-1}\frac{2}{1^2}+\tan^{-1}\frac{2}{2^2}+\tan^{-1}\frac{2}{3^2}+$ &c

5. $\frac{\pi}{4}=\tan^{-1}\frac{1}{2\cdot1^2}+\tan^{-1}\frac{1}{2\cdot2^2}+$ &c $=\tan^{-1}\frac{2}{1^2}-\tan^{-1}\frac{2}{2^2}+$ &c

6. $\frac{\pi}{8}=\tan^{-1}\frac{1}{(1+\sqrt{2})^2}+\tan^{-1}\frac{1}{(1+2\sqrt{2})^2}+\tan^{-1}\frac{1}{(1+3\sqrt{2})^2}+$ &c

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8. If $1 = A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + \&c$:

and $\begin{cases} P_n = A_1 P_{n-1} + A_2 P_{n-2} + A_3 P_{n-3} + \&c \cdots + A_{n-1} P_1. \\ Q_n = A_1 Q_{n-1} + A_2 Q_{n-2} + A_3 Q_{n-3} + \cdots + A_{n-1} Q_1 + A_n Q_0. \\ P_1 = 1 \text{ and } Q_0 = 1 \end{cases}$

then $\frac{P_n}{Q_n}$ approaches x when n becomes greater & greater.

Eg. 1. $x + x^2 = 1$

$x = \frac{0}{1}, \frac{1}{1}, \left| \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \&c \&c \right.$

2. $x + x^2 + x^3 = 1$

$x = \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \left| \frac{2}{4}, \frac{4}{7}, \frac{7}{13}, \frac{13}{24}, \frac{24}{44}, \&c \right.$

3. $x + x^3 = 1$

$x = \frac{0}{1}, \frac{1}{1}, \frac{1}{1}, \left| \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{6}, \frac{6}{9}, \frac{9}{13}, \frac{13}{19}, \&c \right.$

N.B. If $\frac{p}{q}$ and $\frac{r}{s}$ are two consecutive Convergents to x then we may take $\frac{mp + nr}{mq + ns}$ in a Suitable manner equivalent to x.

Ex. 1. Find convergents to $\log_e 2$.

Sol. Let $\log_e 2 = x$ then $e^x = 2$

$\therefore 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \&c$

$\therefore x = \frac{0}{1}, \frac{1}{1}, \left| \frac{1}{1\frac{1}{2}}, \frac{1\frac{1}{2}}{2\frac{1}{6}}, \frac{2\frac{1}{6}}{3\frac{1}{8}}, \frac{3\frac{1}{8}}{4\frac{61}{110}} \right. \&c$

$= \frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{9}{13}, \frac{52}{75}, \frac{375}{541} \&c$

2. When $e^{-x} = x$ shew that the convergents to x are $\frac{1}{2}, \frac{4}{7}, \frac{21}{37}, \frac{148}{261}, \&c.$

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If $\phi(x) = e^x \psi(x)$ then

$$\phi(0) f(0) + \frac{\phi'(0) f'(0)}{\lfloor 1} + \frac{\phi''(0) f''(0)}{\lfloor 2} + \&c$$

$$= \psi(0) f(1) + \frac{\psi'(0) f'(1)}{\lfloor 1} + \frac{\psi''(0) f''(1)}{\lfloor 2} + \&c$$

$$\frac{f(0)}{n} + \frac{f'(0)}{(n+1)\lfloor 1} + \frac{f''(0)}{(n+1)\lfloor 2} + \&c$$

$$= \frac{f(1)}{n} - \frac{f'(1)}{n(n+1)} + \frac{f''(1)}{n(n+1)(n+2)} - \&c$$

$$e^x \left\{ \frac{x}{\lfloor 1} - \frac{2x^2}{2\lfloor 2} + \frac{2^2 x^3}{3\lfloor 3} - \frac{2^3 x^4}{4\lfloor 4} + \&c \right\}$$

$$= \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3}\left(1 + \tfrac{1}{3}\right) + \frac{x^4}{\lfloor 4}\left(1 + \tfrac{1}{3}\right) + \frac{x^5}{\lfloor 5}\left(1 + \tfrac{1}{3} + \tfrac{1}{5}\right)$$

$$+ \frac{x^6}{\lfloor 6}\left(1 + \tfrac{1}{3} + \tfrac{1}{5}\right) + \&c$$

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CHAPTER III

Let $\frac{1^n}{\lfloor 0}x + \frac{2^n}{\lfloor 1}x^2 + \frac{3^n}{\lfloor 2}x^3 + \frac{4^n}{\lfloor 3}x^4 + \&c = e^x f_n(x)$.

1. $\frac{x}{n\lfloor 0} + \frac{x^2}{(n+1)\lfloor 1} + \frac{x^3}{(n+2)\lfloor 2} + \frac{x^4}{(n+3)\lfloor 3} + \frac{x^5}{(n+4)\lfloor 4} + \&c$

$= e^x\left\{\frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c\right\}$

Sol. $L.H.S = \frac{1}{x^{n-1}}\left\{\frac{x^n}{n\lfloor 0} + \frac{x^{n+1}}{(n+1)\lfloor 1} + \frac{x^{n+2}}{(n+2)\lfloor 2} + \&c\right\}$

$= \frac{1}{x^{n-1}}\int x^{n-1}e^x dx$

$= \frac{e^x}{x^{n-1}}\left\{\int x^{n-1}dx - \iint x^{n-1}(dx)^2 + \iiint x^{n-1}(dx)^3 - \&c\right\}$

$= \frac{e^x}{x^{n-1}}\left\{\frac{x^n}{n} - \frac{x^{n+1}}{n(n+1)} + \frac{x^{n+2}}{n(n+1)(n+2)} - \&c\right\} = R.H.S.$

or thus:—

$\frac{1}{n+m} = \frac{1}{n} - \frac{m}{n(n+m)} = \frac{1}{n} - \frac{m}{n(n+1)} + \frac{m(m-1)}{n(n+1)(n+m)}$.

$= \frac{1}{n} - \frac{m}{n(n+1)} + \frac{m(m-1)}{n(n+1)(n+2)} - \frac{m(m-1)(m-2)}{n(n+1)(n+2)(n+m)}$

&c &c.

$\therefore \frac{1}{n+m} = \frac{1}{n} - \frac{m}{n(n+1)} + \frac{m(m-1)}{n(n+1)(n+2)} - \&c$

$\therefore \frac{1}{(n+m)\lfloor m} = \frac{1}{n\lfloor m} - \frac{1}{n(n+1)\lfloor m-1} + \frac{1}{n(n+1)(n+2)\lfloor m-2} - \&c$

But $\frac{1}{(n+m)\lfloor m}$ is the coeff.^{c} of x^{m+1} in L.H.S and the other is that of x^{m+1} in R.H.S. $\therefore$ L.H.S = R.H.S.

Cor. $\frac{x}{\lfloor 1} + \left(1+\frac{1}{2}\right)\frac{x^2}{\lfloor 2} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^3}{\lfloor 3} + \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\frac{x^4}{\lfloor 4} + \&c$

$= e^x\left\{\frac{x}{\lfloor 1} - \frac{x^2}{2\lfloor 2} + \frac{x^3}{3\lfloor 3} - \frac{x^4}{4\lfloor 4} + \&c\right\}$

Sol. $\frac{x}{n} + \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} + \&c = e^x\left\{\frac{x}{n\lfloor 0} - \frac{x^2}{(n+1)\lfloor 1} + \&c\right\}$

Differentiating both Sides with regards to x and then writing 1 for x we can get the result.

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2. $\frac{x}{n+1} - \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} - \&c$

$= \frac{f_0(x)}{n} - \frac{f_1(x)}{n^2} + \frac{f_2(x)}{n^3} - \frac{f_3(x)}{n^4} + \&c$

Sol. By III / we have,

$e^x \left\{ \frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c \right\}$

$= \frac{x}{n\lfloor 0} + \frac{x^2}{(n+1)\lfloor 1} + \frac{x^3}{(n+2)\lfloor 2} + \&c$

changing n to n+1, we have

$e^x \left\{ \frac{x}{n+1} - \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} - \&c \right\}$

$= \frac{1}{n} \left\{ \frac{x}{\lfloor 0} + \frac{x^2}{\lfloor 1} + \frac{x^3}{\lfloor 2} + \frac{x^4}{\lfloor 3} + \&c \right\}$

$- \frac{1}{n^2} \left\{ \frac{1}{\lfloor 0} x + \frac{2}{\lfloor 1} x^2 + \frac{3}{\lfloor 2} x^3 + \frac{4}{\lfloor 3} x^4 + \&c \right\}$

$+ \frac{1}{n^3} \left\{ \frac{1^2}{\lfloor 0} x + \frac{2^2}{\lfloor 1} x^2 + \frac{3^2}{\lfloor 2} x^3 + \frac{4^2}{\lfloor 3} x^4 + \&c \right\}$

$- \&c = \frac{e^x}{n} f_0(x) - \frac{e^x}{n^2} f_1(x) + \frac{e^x}{n^3} f_2(x) - \&c$ by our Supposition. $\therefore$ L.H.S = R.H.S.

N.B. An easier solution for III / is as follows —

Let $\phi(n) = \frac{x}{n\lfloor 0} + \frac{x^2}{(n+1)\lfloor 1} + \frac{x^3}{(n+2)\lfloor 2} + \&c$

Then $n\phi(n) = x + \frac{n}{n+1} \frac{x^2}{\lfloor 1} + \frac{n}{n+2} \frac{x^3}{\lfloor 2} + \&c$

and $x\phi(n+1) = \frac{x^2}{n+1} + \frac{x^3}{(n+2)\lfloor 1} + \&c$

$\therefore n\phi(n) + x\phi(n+1) = x + x^2 + \frac{x^3}{\lfloor 2} + \frac{x^4}{\lfloor 3} + \&c = xe^x$

$\therefore \phi(n) = e^x \frac{x}{n} - \frac{x}{n} \phi(n+1) = e^x \cdot \frac{x}{n} - e^x \cdot \frac{x^2}{n(n+1)} + \frac{x^2}{n(n+1)} \phi(n+2)$

&c &c.

$\therefore \frac{x}{n\lfloor 0} + \frac{x^2}{(n+1)\lfloor 1} + \frac{x^3}{(n+2)\lfloor 2} + \frac{x^4}{(n+3)\lfloor 3} + \&c$

$= e^x \left\{ \frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c \right\}$

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3. $e^{x(e^a-1)} = 1 + \frac{a}{\lfloor 1} f_0(x) + \frac{a^2}{\lfloor 2} f_1(x) + \frac{a^3}{\lfloor 3} f_2(x) + \&c$

Sol. $e^{xe^a} = 1 + xe^a + \frac{x^2}{\lfloor 2} e^{2a} + \frac{x^3}{\lfloor 3} e^{3a} + \&c.$

The coeff^{t} of a^{n+1} is $\frac{1}{\lfloor n+1} \left\{ \cdot \frac{1^n}{\lfloor 0} x + \frac{2^n}{\lfloor 1} x^2 + \frac{3^n}{\lfloor 2} x^3 + \&c \right\}$

$= \frac{e^x}{\lfloor n+1} f_n(x).$

$\therefore e^{xe^a} = e^x \left\{ 1 + \frac{a}{\lfloor 1} f_0(x) + \frac{a^2}{\lfloor 2} f_1(x) + \frac{a^3}{\lfloor 3} f_2(x) + \&c \right\}.$

4. $f_n(x) = x \left\{ 1 + n f_0(x) + \frac{n(n-1)}{\lfloor 2} f_1(x) + \frac{n(n-1)(n-2)}{\lfloor 3} f_2(x) + \&c \right\}$

Sol. Differentiating both sides in III 3 with regards to a we have $xe^a e^{x(e^a-1)} = f_0(x) + \frac{a}{\lfloor 1} f_1(x) + \frac{a^2}{\lfloor 2} f_2(x) + \&c$

But $xe^a e^{x(e^a-1)} = xe^a \left\{ 1 + \frac{a}{\lfloor 1} f_0(x) + \frac{a^2}{\lfloor 2} f_1(x) + \&c \right\}$

Equating the coeff^{ts} of a^n we get the result.

N. B. The above result may be written thus —

$f_0(x), f_1(x), f_2(x), f_3(x), f_4(x)$

$a_0 \quad b_0 \quad c_0 \quad d_0$

$a_1 \quad b_1 \quad c_1$

$a_2 \quad b_2$

a_3

} These are successive diff^{s} a_n being equal to $x f_n(x)$.

5. If $f_n(x) = \phi_1(n)\, x + \phi_2(n)\, x^2 + \phi_3(n)\, x^3 + \dots + \phi_{n+1}(n)\, x^{n+1}$

then $\frac{\phi_n(n)}{\lfloor 0} + \frac{\phi_{n-1}(n)}{\lfloor 1} + \frac{\phi_{n-2}(n)}{\lfloor 2} + \&c = \frac{n^n}{\lfloor n-1}$

Sol. $e^x f_n(x) = e^x \left\{ \phi_1(n)\, x + \phi_2(n)\, x^2 + \dots + \phi_{n+1}(n)\, x^{n+1} \right\}$

Again we have $e^x f_n(x) = \frac{1^n}{\lfloor 0} x + \frac{2^n}{\lfloor 1} x^2 + \frac{3^n}{\lfloor 2} x^3 + \&c$

By equating the coeff^{ts} of x^n in both sides we can get the result.

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$$\phi(0) + \frac{\phi(1)}{\lfloor 1} x + \frac{\phi(2)}{\lfloor 2} x^2 + \cdots$$

$$= e^x \left\{ \phi(x) + \frac{x}{2} \phi'(x) + \cdots \right\} \text{ nearly}$$

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6. $\phi_{n+1}(n)\,\lfloor n = (n+1)^n - n\cdot n^n + \frac{n(n-1)}{\lfloor 2}(n-1)^n - \frac{n(n-1)(n-2)}{\lfloor 3}(n-2)^n$
$+ \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}(n-3)^n - \&c$

Sol. $f_n(x) = \phi_1(n)\,x + \phi_2(n)\,x^2 + \phi_3(n)\,x^3 + \&c$
$= e^{-x}\left\{\frac{1^n}{\lfloor 0}x + \frac{2^n}{\lfloor 1}x^2 + \frac{3^n}{\lfloor 2}x^3 + \&c\right\}$.

By equating the coeffts of x^{n+1} we can get the result.

7. $\phi_n(n+1) = n\,\phi_n(n) + \phi_{n-1}(n)$.

Sol. $\phi_n(n+1) = \frac{1}{\lfloor n-1}\left\{n^{n+1} - (n-1)(n-1)^{n+1} + \frac{(n-1)(n-2)}{\lfloor 2}(n-2)^{n+1} - \&c\right\}$

$\therefore \phi_n(n+1) - \phi_{n-1}(n) = \frac{1}{\lfloor n-1}\left\{n\cdot n^n - n(n-1)(n-1)^n + \frac{n(n-1)(n-2)}{\lfloor 2}(n-2)^n - \&c\right\}$

$= n\,\phi_n(n)$. $\therefore \phi_n(n+1) = n\,\phi_n(n) + \phi_{n-1}(n)$. or in words the

$f_0(x) = x$ Write under each term the product of

$f_1(x) = x + x^2$ the coeff^{t} and the index of x of that

$f_2(x) = x + 3x^2 + x^3$ term together with the coeff^{t} of the

$f_3(x) = x + 7x^2 + 6x^3 + x^4$ preceding one.

$f_4(x) = x + 15x^2 + 25x^3 + 10x^4 + x^5$

$f_5(x) = x + 31x^2 + 90x^3 + 65x^4 + 15x^5 + x^6$

$f_6(x) = x + 63x^2 + 301x^3 + 350x^4 + 140x^5 + 21x^6 + x^7$

Ex. 1. If $\frac{a_1}{n+1} - \frac{a_2}{(n+1)(n+2)} + \frac{a_3}{(n+1)(n+2)(n+3)} - \&c$
$= \frac{F(0)}{n} - \frac{F(1)}{n^2} + \frac{F(2)}{n^3} - \frac{F(3)}{n^4} + \&c$

shew that $F(n) = \phi_1(n)\,a_1 + \phi_2(n)\,a_2 + \phi_3(n)\,a_3 + \&c$

2. Shew that $\phi_{n+1}(n)$ is the coeff^{t} of $\frac{x^n}{\lfloor n}$ in $\frac{e^x}{\lfloor n}(e^x-1)^n$.

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Sol. From III 6 we have $\phi_{n+1}(n)\lfloor n$

$= (n+1)^n - \frac{n}{\lfloor 1}n^n + \frac{n(n-1)}{\lfloor 2}(n-1)^n$ &c

= the coeff^{t}. of $\frac{x^n}{\lfloor n}$ in $\left\{e^{x(n+1)} - \frac{n}{\lfloor 1}e^{xn} + \frac{n(n-1)}{\lfloor 2}e^{x(n-1)} \text{ &c}\right\}$

= the coeff^{t}. of $\frac{x^n}{\lfloor n}$ in $e^x(e^x-1)^n$.

3. $\frac{d f_{n-1}(x)}{dx} = n f_{n-2}(x) + \frac{n(n-1)}{\lfloor 2} f_{n-3}(x) + \frac{n(n-1)(n-2)}{\lfloor 3} f_{n-4}(x) + \text{&c}$

Sol. Differentiating both Sides in III 3 with regards to x and then differentiating the result with regards to a and then equating the coeff^{ts} as in III 4. we can get the result.

4. $\frac{1}{2}f_n(x) + \int f_n(x)\,dx = \frac{f_{n+1}(x)}{n+1} + \frac{B_2}{\lfloor 2} n f_{n-1}(x)$

$- \frac{B_4}{\lfloor 4} n(n-1)(n-2) f_{n-3}(x) + \frac{B_6}{\lfloor 6} n(n-1)(n-2)(n-3)(n-4) f_{n-5}(x)$

$-$ &c

Sol. Integrating both Sides in III 3. with regards to x we have $\frac{1}{e^a-1}\left\{1 + \frac{a}{\lfloor 1}f_0(x) + \frac{a^2}{\lfloor 2}f_1(x) + \frac{a^3}{\lfloor 3}f_2(x) + \text{&c}\right\}$

$= \frac{1}{e^a-1} + x + \frac{a}{\lfloor 1}\int f_0(x)\,dx + \frac{a^2}{\lfloor 2}\int f_1(x)\,dx + \frac{a^3}{\lfloor 3}\int f_2(x)\,dx + \text{&c}$

Equating the coeff^{ts} of a^{n+1} in both Sides we can get the results.

5. If $\frac{1^n}{\lfloor 0} + \frac{2^n}{\lfloor 1} + \frac{3^n}{\lfloor 2} + \frac{4^n}{\lfloor 3} + \text{&c} = eA_n$

shew that $A_0 = 1$, $A_1 = 2$, $A_2 = 5$, $A_3 = 15$, $A_4 = 52$, $A_5 = 203$, $A_6 = 877$, $A_7 = 4140$, $A_8 = 21147$ &c

Sol. $2 = 1+1$, $5 = 1+2\cdot1+2$, $15 = 1+3\cdot1+3\cdot2+5$, $52 = 1+4\cdot1+6\cdot2 + 4\cdot5+15$, $203 = 1+5\cdot1+10\cdot2+10\cdot5+5\cdot15+52$ &c

6. If $\frac{1^n}{\lfloor 0} + \frac{2^n}{\lfloor 1} - \frac{3^n}{\lfloor 2} + \frac{4^n}{\lfloor 3} - \text{&c} = \frac{A_n}{e}$, shew that $A_0 = -1$, $A_1 = 0$, $A_2 = 1$, $A_3 = 1$, $A_4 = -2$, $A_5 = -9$, $A_6 = -9$, $A_7 = 50$

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$A_8 = 267$ &c &c

N.B.

1	2	5	15	52	203	877		-1	0	1	1	-2	-9	-9	50
	1	3	10	37	151	674			1	1	0	-3	-7	0	59
		2	7	17	114	523				0	-1	-3	-4	7	59
			5	20	87	409					-1	-2	-1	11	52
				15	67	322						-1	1	12	41
					52	255							2	11	29
						203								9	18
															9

7 Shew that

i. $\frac{1^3}{\lfloor 0} + \frac{2^3}{\lfloor 1} + \frac{3^3}{\lfloor 2} + \frac{4^3}{\lfloor 3} + \&c = 3\left(\frac{1^2}{\lfloor 0} + \frac{2^2}{\lfloor 1} + \frac{3^2}{\lfloor 2} + \&c\right)$

ii. $\frac{1^2(1^3+1)}{\lfloor 0} + \frac{2^2(2^3+1)}{\lfloor 1} + \frac{3^2(3^3+1)}{\lfloor 2} + \&c = 4\left(\frac{1^4}{\lfloor 0} + \frac{2^4}{\lfloor 1} + \frac{3^4}{\lfloor 2} + \&c\right)$

iii. $\frac{1^3}{\lfloor 0} - \frac{2^3}{\lfloor 1} + \frac{3^3}{\lfloor 2} - \frac{4^3}{\lfloor 3} + \&c = \frac{1^2}{\lfloor 0} - \frac{2^2}{\lfloor 1} + \frac{3^2}{\lfloor 2} - \frac{4^2}{\lfloor 3} + \&c$

iv. $\frac{1^6}{\lfloor 0} - \frac{2^6}{\lfloor 1} + \frac{3^6}{\lfloor 2} - \frac{4^6}{\lfloor 3} + \&c = \frac{1^5}{\lfloor 0} - \frac{2^5}{\lfloor 1} + \frac{3^5}{\lfloor 2} - \frac{4^5}{\lfloor 3} + \&c$

v. $\frac{1^3(1^2+1)(1^3+1)}{\lfloor 0} - \frac{2^3(2^2+1)(2^3+1)}{\lfloor 1} + \frac{3^3(3^2+1)(3^3+1)}{\lfloor 2} - \&c$

$= 5\left(\frac{1^7}{\lfloor 0} - \frac{2^7}{\lfloor 1} + \frac{3^7}{\lfloor 2} - \frac{4^7}{\lfloor 3} + \frac{5^7}{\lfloor 4} - \&c\right)$.

8. If $\frac{x}{n+a+b} - \frac{bx^2}{(n+a+b)(n+a+2b)} + \frac{b^2x^3}{(n+a+b)(n+a+2b)(n+a+3b)}$

$- \&c = \frac{F_0(x)}{n} - \frac{F_1(x)}{n^2} + \frac{F_2(x)}{n^3} - \frac{F_3(x)}{n^4} + \&c$, then

$(a+b)^n \frac{x}{\lfloor 0} + (a+2b)^n \frac{x^2}{\lfloor 1} + (a+3b)^n \frac{x^3}{\lfloor 2} + \&c = e^x F_n(x)$.

Cor 1. $F_0(x) + y F_1(x) + \frac{y^2}{\lfloor 1} F_2(x) + \frac{y^3}{\lfloor 3} F_3(x) + \&c = x e^{y(a+b)} e^{x(e^{by}-1)}$

Cor 2. $F_{n+1}(x) - (a+b) F_n(x) = bx\left\{F_n(x) + \frac{n}{\lfloor 1} F_{n-1}(x) b + \frac{n(n-1)}{\lfloor 2} F_{n-2}(x) b^2 + \&c\right\}$.

Cor 3. If $F_n(x) = \phi_1(n) x + \phi_2(n) x^2 + \phi_3(n) x^3 + \&c$, then

$\frac{\phi_n(n)}{\lfloor 0} + \frac{\phi_{n-1}(n)}{\lfloor 1} + \frac{\phi_{n-2}(n)}{\lfloor 2} + \&c = \frac{(a+nb)^n}{\lfloor n-1}$.

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N.B. If $F_{n+1}(x) - (a+b)F_n(x) = \psi_n(x)$, then

$\psi_0(x)$	$\psi_1(x)$	$\psi_2(x)$	$\psi_3(x)$	$\psi_4(x)$	$\psi_5(x)$	
	a_1	b_1	c_1	d_1	e_1	These are Successive diff^{a}. b times the previous term being Subtracted from each term and a_n being equal to $b \times F_n(x)$.
		a_2	b_2	c_2	d_2	
			a_3	b_3	c_3	
				a_4	b_4	
					a_5	

Cor 4. $\phi_n(n)\,\lfloor n-1 = (a+nb)^n - \frac{n-1}{\lfloor 1}(a+n-1b)^n + \frac{(n-1)(n-2)}{\lfloor 2}(a+n-2b)^n$

$- \frac{(n-1)(n-2)(n-3)}{\lfloor 3}(a+n-3b)^n + \&c$

Cor 5. $\phi_n(n+1) = (a+nb)\,\phi_n(n) + b\,\phi_{n-1}(n)$ or in words thus —

Write under each term the product of $a+nb$, n being the index of x, and the coeff^{t}. of x of that term together with b times the coeff^{t}. of the preceding one.

$F_0(x) = x$

$F_1(x) = (a+b)x + bx^2$

$F_2(x) = (a+b)^2x + b(2a+3b)x^2 + b^2x^3$

$F_3(x) = (a+b)^3x + b\{3(a+b)(a+2b) + b^2\}x^2 + 3b^2(a+2b)x^3 + b^3x^4$

$F_4(x) = (a+b)^4x + b\{2(a+b)(a+2b) + b^2\}(2a+3b)x^2 + b^2\{6(a+2b)^2 + b^2\}$

$+ x^3 + 2b^3(2a+5b)x^4 + b^4x^5$.

Ex. 1. Shew that $\phi_{n+1}(n)$ is the coeff^{t}. of $\frac{x^n}{\lfloor n}$ in $\frac{e^{x(a+b)}}{\lfloor n}(e^{bx}-1)^n$.

2 Shew that i. $\frac{1^3+1^2}{\lfloor 0} - \frac{3^3+3^2}{\lfloor 1} + \frac{5^3+5^2}{\lfloor 2} - \&c = 0$

ii. $\frac{1^4}{\lfloor 0} + \frac{2^4}{\lfloor 1} + \frac{3^4}{\lfloor 2} + \frac{4^4}{\lfloor 3} + \&c = 4\left(\frac{1^2}{\lfloor 0} + \frac{3^2}{\lfloor 1} + \frac{5^2}{\lfloor 2} + \&c\right)$

iii. $\frac{1^7+1^6}{\lfloor 0} - \frac{2^7+2^6}{\lfloor 1} + \frac{3^7+3^6}{\lfloor 2} - \&c = \frac{1^4}{\lfloor 0} - \frac{3^4}{\lfloor 1} + \frac{5^4}{\lfloor 2} - \frac{7^4}{\lfloor 3} + \&c$.

iv. $1^4 - \frac{3^4}{\lfloor 1} + \frac{5^4}{2\lfloor 2} - \frac{7^4}{3\lfloor 3} + \&c = \left(1 - \frac{1}{\lfloor 1} + \frac{1}{2\lfloor 2} - \frac{1}{3\lfloor 3} + \&c\right) - 4$.

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CHAPTER IV

1. If $x^n + \frac{(n+1)^{n+1}}{a\lfloor 1} + \frac{(n+2)^{n+2}}{a^2\lfloor 2} + \frac{(n+3)^{n+3}}{a^3\lfloor 3} + \&c = F_n(x)$, then

$$F_{n+1}(x) = x\,F_n(x) + \frac{1}{a}F_{n+1}(n+1).$$

Sol. $F_{n+1}(x) = x^{n+1} + \frac{(n+1)^{n+2}}{a\lfloor 1} + \frac{(n+2)^{n+3}}{a^2\lfloor 2} + \&c$

$= x\left\{x^n + \frac{(n+1)^{n+1}}{a\lfloor 1} + \frac{(n+2)^{n+2}}{a^2\lfloor 2} + \&c\right\} + \frac{1}{a}\left\{(n+1)^{n+1} + \frac{(n+2)^{n+2}}{a\lfloor 1} + \&c\right\}$

$= x\,F_n(x) + \frac{1}{a}F_{n+1}(n+1)$.

We see from this identity that if we are able to find the sum for one value of n we can sum up the series for all values of n.

2. If $x = a\log_e x$, then $\frac{x^n}{n} = n^{-1} + \frac{(n+1)^0}{a\lfloor 1} + \frac{(n+2)^1}{a^2\lfloor 2} + \frac{(n+3)^2}{a^3\lfloor 3} + \&c$

Sol. Suppose $f(n) = 1 + \frac{n}{a\lfloor 1} + \frac{n(n+1)}{a^2\lfloor 2} + \frac{n(n+3)^2}{a^3\lfloor 3} + \&c$.

If we multiply $f(n)$ by $f(m)$ we get $f(m+n)$.

$\therefore f(n) = \{f(1)\}^n$. Let $f(1) = x$ then $x^n = f(n)$.

$\frac{f(n)-1}{n}$, when $n=0$, $= \frac{1}{a} + \frac{1}{a^2\lfloor 2} + \frac{3^2}{a^3\lfloor 3} + \&c$

$= \frac{1}{a}\left(1 + \frac{1}{a\lfloor 1} + \frac{3}{a^2\lfloor 2} + \frac{4^2}{a^3\lfloor 3} + \&c\right) = \frac{1}{a}f(1) = \frac{x}{a}$.

i.e $\frac{x^n-1}{n}$ when $n=0$, $= \frac{x}{a}$ or $\log_e x = \frac{x}{a}$ or $x = a\log_e x$.

N.B. The minimum value of $\frac{x}{\log_e x}$ is e. If $a = e$ $f(n) = e^n$.

If $a > e$ it is convergent but if $a < e$ it is divergent.

Cor. $e^x = 1 + \frac{x}{e^{\lfloor 1}} + \frac{x(x+2n)}{e^{2n}\lfloor 2} + \frac{x(x+3n)^2}{e^{3n}\lfloor 3} + \frac{x(x+4n)^3}{e^{4n}\lfloor 4} + \&c$

Sol. Write e^x for x and $\frac{n}{x}$ for n in IV 2.

N.B. In a similar manner we can prove that

if $a\,q\,x^p - x^q + 1 = 0$, then

$$x^n = 1 + \frac{n}{\lfloor 1}a + \frac{n(n+2p-q)}{\lfloor 2}a^2 + \frac{n(n+3p-q)(n+3p-2q)}{\lfloor 3}a^3$$
$$+ \frac{n(n+4p-q)(n+4p-2q)(n+4p-3q)}{\lfloor 4}a^4 + \&c.$$

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Ex. 1. Shew that

$$2^m = 1 + \frac{m}{2^n} + \frac{m(m+2n-1)}{2^{2n}\lfloor 2} + \frac{m(m+3n-1)(m+3n-2)}{2^{3n}\lfloor 3} + \&c.$$

2. $$\left(\frac{2}{1+\sqrt{1-4x}}\right)^n = 1 + nx + \frac{n(n+3)}{\lfloor 2}x^2 + \frac{n(n+4)(n+5)}{\lfloor 3}x^3 + \&c$$

3. Expand x in terms of a in each of the following.

i. $x^a = e^{\pm x}$; Sol. $a \log_e x = \pm x$.

ii. $x^a = a^{\pm x}$; Sol. $a \log_e x = \pm x \log_e a$. $\therefore \frac{x}{\log_e x} = \pm \frac{a}{\log_e a}$.

iii. $x = a e^{\pm x}$; Sol. Let $x = \log_e y$, then $\log_e y = a y^{\pm 1}$

iv. $x = a^{\pm x}$; Sol. Let $x \log_e a = \log_e y$, then $\log_e y = y^{\pm 1} \log_e a$.

v. $x^{\pm x} = a$; Sol. Let $x = \frac{1}{y}$ then $y = a^{\mp y}$

vi. $x e^{\pm x} = a$; Sol. Let $x = \log_e y$ then $\log_e y = a y^{\pm 1}$

vii. $e^x \pm x = u$; Sol. Let $x = \log_e \log_e y$ then $e^a = y (\log_e y)^{\pm 1}$

viii. $x \pm \log_e x = a$; Sol. Let $x = \log_e y$ then $e^a = y (\log_e y)^{\pm 1}$

4. Shew how to find the values of the following for special values of x

i. $x^{x^{x \&c}}$; Sol. Let its value be equal to V. then $x^V = V$.

ii. $x \pm e^{x \pm e^{x \pm \&c}}$

Sol. $x \pm e^V = V$.

iii. $\log_e \left\{ x \log_e \left[x \log_e (x \ldots \text{ad inf.}) \right] \right\}$

iv. $\pm \log_e \left\{ x \pm \log_e \left[x \pm \log_e (x \pm \&c) \right] \right\}$

3. If we write $x^n \phi_n(n)$ for $F_n(n)$, we see that

$$\phi_{n+1}(n) - \log_e x \, \phi(n+1, n+1) = n \phi(n, n).$$

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If n is positive

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Case I If n is positive

Let $\phi_n(x) = \frac{\psi_1(x,n)}{(1-\log_e x)^{n+1}} + \frac{\psi_2(x,n)}{(1-\log_e x)^{n+2}} + \cdots + \frac{\psi_{n+1}(x,n)}{(1-\log_e x)^{2n+1}}$.

Then by IV. 1 we see that

$$n\,\psi_t(x,n) + \psi_{t-1}(x+1, n+1) = \psi_t(x+1, n+1) + \psi_{t-1}(x+1, n).$$

Case II If n is negative the terms in R.H.S continue as far as the term independent of $(1-\log_e x)$.

N.B $F_n(x)$ is convergent when $a > e$ or $< e$ according as n is positive or not.

Cor. 1. $\psi_1(x,n) + \psi_2(x,n) + \psi_3(x,n)$ + &c as far as the terms cease to continue in $\phi_n(n) = n^n$.

Sol. L.H.S = $\phi_n(n)$ when $x = 1$ i.e $F_n(x)$ when $x = 1$ i.e $F_n(x)$ when $a = \infty$ i.e ~~∞~~.

Cor 2. $e^x = (1-n)\left\{1 + \frac{x+n}{e^n \lfloor 1} + \frac{(x+2n)^2}{e^{2n} \lfloor 2} + \frac{(x+3n)^3}{e^{3n} \lfloor 3} + \&c\right\}$

N.B. $\phi(-1, n) = \frac{1}{n}$

$$\phi(-2, n) = \frac{1-\log_e x}{n(n+1)} + \frac{1}{n^2(n+1)}.$$

$$\phi(-3, n) = \frac{(1-\log_e x)^2}{n(n+1)(n+2)} + \frac{(3n+2)(1-\log_e x)}{n^2(n+1)^2(n+2)} + \frac{3n+2}{n^3(n+1)^2(n+2)}$$

$$\phi(0, n) = \frac{1}{1-\log_e x}.$$

$$\phi(1, n) = \frac{n-1}{(1-\log_e x)^2} + \frac{1}{(1-\log_e x)^3}$$

$$\phi(2, n) = \frac{(n-1)(n-2)}{(1-\log_e x)^3} + \frac{(n-1)(n-2)\left(\frac{1}{n-2} + \frac{2}{n-1}\right)}{(1-\log_e x)^4} + \frac{1.3}{(1-\log_e x)^5}$$

$$\phi(3, n) = \frac{(n-1)(n-2)(n-3)}{(1-\log_e x)^4} + \frac{(n-1)(n-2)(n-3)\left(\frac{1}{n-3} + \frac{2}{n-2} + \frac{3}{n-1}\right)}{(1-\log_e x)^5}$$

$$+ \frac{15n-35}{(1-\log_e x)^6} + \frac{1.3.5}{(1-\log_e x)^7}.$$

4. To expand x^m in ascending powers of h when

$$x^x = a^a e^h.$$

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Let $\frac{x-a}{a} = \frac{A_1}{\lfloor 1}\cdot\frac{h}{a} - \frac{A_2}{\lfloor 2}\left(\frac{h}{a}\right)^2 + \frac{A_3}{\lfloor 3}\left(\frac{h}{a}\right)^3 - \&c$

and let $n = \frac{1}{1+\log_e a}$.

then $A_n - n(n-2)A_{n-1} = n\left\{2A_1A_{n-1} + \frac{n(n-1)}{\lfloor 2}A_2A_{n-2} + \right.$

$\left.\frac{n(n-1)(n-2)}{\lfloor 3}A_3A_{n-3} + \&c\right\}$ the last term being

$\frac{\lfloor n}{\lfloor\frac{n-1}{2}\lfloor\frac{n+1}{2}}A_{\frac{n-1}{2}}$ or $\frac{\lfloor n}{2\left(\lfloor\frac{n}{2}\right)^2}A_{\frac{n}{2}}$ according as n is odd or even.

Multiply the power and the coeff^{t}. Write under it the sum of this product and $(n-1)$ times the coeff^{t}. of the preceding term where n is the suffix of A.

$A_1 = n$

$A_2 = n^3$

$A_3 = 3n^5 + n^4$

$A_4 = 15n^7 + 10n^6 + 2n^5$

$A_5 = 105n^9 + 105n^8 + 40n^7 + 6n^6$

$A_6 = 945n^{11} + 1260n^{10} + 700n^9 + 196n^8 + 24n^7$

$A_7 = 10395n^{13} + 17325n^{12} + 12600n^{11} + 5068n^{10} + 1148n^9 + 120n^8$

N.B. For $\frac{a}{x}$ take $n+1$ times the coeff^{t}.; for $\log_e\frac{x}{a}$ take n times the coeff^{t}. and generally for $\frac{x^m}{a^m}$ take $n-m$ times the coeff^{t}.

Ex. 1. Shew that the sum of the coeff^{ts}. of A_n is $(n-1)^{n-1}$

Sol. Put 1 for a. Then $x^x = e^h$. Let $x = \frac{1}{y}$

then $y^{\frac{1}{y}} = e^{-h}$ or $\frac{1}{y}\log_e y = -h$

∴ By applying III 2 we have $\frac{1}{y} = x = 1 + h - \frac{1^1}{\lfloor 2}h^2 + \frac{2^2}{\lfloor 3}h^3$

$- \frac{3^3}{\lfloor 4}h^4 + \&c$. ∴ The sum of the coeff^{ts}. of $A_n = (n-1)^{n-1}$.

2. Expand x in ascending powers of h when $\sqrt[x]{x} = e^h\sqrt[a]{a}$

Let $x = \frac{1}{y}$ then $y^y = e^{-h}\left(\frac{1}{a}\right)^{\frac{1}{a}}$.

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CHAPTER V

Let $F_1(x) = e^x - 1$, $F_2(x) = e^{e^x - 1} - 1$, $F_3(x) = e^{e^{e^x - 1} - 1} - 1$, $F_4(x) = e^{e^{e^{e^x - 1} - 1} - 1} - 1$, $F_5(x) = e^{e^{e^{e^{e^x - 1} - 1} - 1} - 1} - 1$ &c &c

Now let us try to find the expansion of $F_n(x)$

(1) In ascending powers of x
(2) In ascending powers of n

1. Let $e^{e^{\cdot^{\cdot^{\cdot^{e^x - 1} - 1} - 1} - 1} - 1} - 1 = F_n(x) = x\phi_1(n) + x^2\phi_2(n) + x^3\phi_3(n) + \&c$

$= f_0(x) + n f_1(x) + n^2 f_2(x) + n^3 f_3(x) + \&c$

then $\log_e\left[1 + \log_e\left\{1 + \log_e\left(1 + \cdots + \log_e(1+x)\right)\right\}\right]$ logarithms being taken n times

$= F_{-n}(x) = x\phi_1(-n) + x^2\phi_2(-n) + x^3\phi_3(-n) + \&c$

$= f_0(x) - n f_1(x) + n^2 f_2(x) - n^3 f_3(x) + \&c$

Sol. We have $e^{F_{n-1}(x)} - 1 = F_n(x)$. $\therefore F_{n-1}(x) = \log_e\{1 + F_n(x)\}$

$\therefore F_0(x) = x$. $\therefore F_{-1}(x) = \log_e(1+x)$, $\therefore F_{-2}(x) = \log_e\{1 + \log_e(1+x)\}$ &c &c

Cor. $F_0(x) = x$ and $f_0(x) = x$.

2. $\dfrac{d F_n(x)}{dx} \div \dfrac{d F_{n-1}(x)}{dx} = 1 + F_n(x)$.

Sol. $e^{F_{n-1}(x)} - 1 = F_n(x)$. $\therefore F_{n-1}(x) = \log_e\{1 + F_n(x)\}$

Differentiating both sides with regards to x we have

$\dfrac{d F_n(x)}{dx} = \{1 + F_n(x)\} \dfrac{d F_{n-1}(x)}{dx}$.

Cor. 1. $\dfrac{d F_n(x)}{dx} = \{1 + F_1(x)\}\{1 + F_2(x)\}\{1 + F_3(x)\} \cdots \{1 + F_n(x)\}$.

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Sol. From IV 2 we have $F_n'(x) = \{1 + F_n(x)\} F_{n-1}'(x)$.

$= \{1 + F_n(x)\}\{1 + F_{n-1}(x)\} F_{n-2}'(x) = \{1 + F_n(x)\}\{1 + F_{n-1}(x)\}\{1 + F_{n-2}(x)\} F_{n-3}'(x)$.

$= \&c\ \&c = \{1 + F_n(x)\}\{1 + F_{n-1}(x)\}\{1 + F_{n-2}(x)\} \cdots \{1 + F_1(x)\} F_0'(x)$.

But from V 1 Cor $F_0(x) = x \quad \therefore F_0'(x) = 1$

Cor 2. $n\{\phi_n(x) - \phi_n(n-1)\} = (n-1)\phi_1(x)\phi_{n-1}(n-1) + (n-2)\phi_2(x)\phi_{n-2}(n-1)$

$+ (n-3)\phi_3(x)\phi_{n-3}(n-1) + \&c$.

Sol. $F_{n-1}'(x)\{1 + F_n(x)\} = F_n'(x)$. Here equate the coeff^{ts} of x^{n+1}.

Cor 3. $\dfrac{d f_1(x)}{dx} = x - \frac{1}{2} f_1(x) + B_2 f_2(x) - B_4 f_4(x) + B_6 f_6(x) - \&c$

Sol. From IV 2 Cor 1 we have $F_n'(x) = \{1 + F_1(x)\}\{1 + F_2(x)\} \cdots \{1 + F_n(x)\}$

i.e $1 + n \dfrac{d f_1(x)}{dx} + \&c = e^{F_0(x) + F_1(x) + \cdots + F_{n-1}(x)}$

$\therefore \log_e\left\{1 + n \dfrac{d f_1(x)}{dx} + \&c\right\} = F_0(x) + F_1(x) + F_2(x) + \&c$ to n terms.

$= \psi(x) + \displaystyle\int_0^n F_n(x)\, dn - \frac{1}{2} F_n(x) + \frac{B_2}{L2} \frac{d F_n(x)}{dn} - \frac{B_4}{L4} \frac{d^3 F_n(x)}{dn^3} + \&c$

where $\psi(x)$ is a function of x independent of n.

Equating the coeff^{ts} of n, we have

$\dfrac{d f_1(x)}{dx} = x - \frac{1}{2} f_1(x) + B_2 f_2(x) - B_4 f_4(x) + B_6 f_6(x) - \&$.

N. B. $\psi(x) = \displaystyle\int_0^x \frac{x - \frac{d f_1(x)}{dx}}{f_1(x)}\, dx$

Sol. Since when $n = 0$, $\log_e\left\{1 + n \dfrac{d f_n}{dx} + \&c\right\} = 0$

$\psi(x) = \dfrac{x}{2} - \dfrac{B_2}{2} f_1(x) + \dfrac{B_4}{4} f_3(x) - \dfrac{B_6}{6} f_5(x) + \&c$

$\therefore \psi'(x) = \dfrac{1}{2} - \dfrac{B_2}{2} f_1'(x) + \dfrac{B_4}{4} f_3'(x) - \dfrac{B_6}{6} f_5'(x) + \&c$.

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$$\left\{1+2\left(\frac{\cos\theta}{\cosh\pi}+\frac{\cos 2\theta}{\cosh 2\pi}+\frac{\cos 3\theta}{\cosh 3\pi}+\&c\right)\right\}^{-2}$$

$$+\left\{1+2\left(\frac{\cosh\theta}{\cosh\pi}+\frac{\cosh 2\theta}{\cosh 2\pi}+\frac{\cosh 3\theta}{\cosh 3\pi}+\&c\right)\right\}^{-2}$$

~~is Constant~~ $= \frac{2}{\pi}(1-\epsilon)^4$.

35. $(\sqrt{5}-2)^8\left(\frac{\sqrt{4+\sqrt{7}}-\sqrt[4]{7}}{2}\right)^{36}(6-\sqrt{35})^6$

$$\times\left(\sqrt{\frac{43+15\sqrt{7}+(8+3\sqrt{7})\sqrt{10\sqrt{7}}}{8}}\right.$$

$$\left.\pm\sqrt{\frac{35+15\sqrt{7}+(8+3\sqrt{7})\sqrt{10\sqrt{7}}}{8}}\right)^{24}$$

21. $\left(\frac{\sqrt{4+\sqrt{7}}-\sqrt[4]{7}}{2}\right)^{24}\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{12}(2-\sqrt{3})^4$

$$\times\left(\frac{\sqrt{3+\sqrt{7}}-\sqrt[4]{8\sqrt{7}}}{\sqrt{3+\sqrt{7}}+\sqrt[4]{8\sqrt{7}}}\right)^{12}$$

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$\therefore \psi'(x) f_1(x) = \frac{1}{2} f_1(x) - \frac{B_2}{2} f_1(x) f_1'(x) + \frac{B_4}{4} f_1(x) f_3'(x) - \&c$

$= \frac{1}{2} f_1(x) - B_2 f_2(x) + B_4 f_4(x) - \&c$ by IV 3.

$= x - f_1'(x)$ by IV 2 Cor 3.

$\therefore \psi'(x) = \frac{x - f_1'(x)}{f_1(x)}$. $\therefore \psi(x) = \int_0^x \frac{x - f_1'(x)}{f_1(x)} dx$.

3. $\frac{d F_n(x)}{dx} = f_1(x) \frac{d F_n(x)}{dx}$ and consequently $n f_n(x) = f_1(x) \frac{d f_{n-1}(x)}{dx}$

Sol. In IV 1. Write $F_K(x)$ for x; then $F_{n+K}(x) = F_n\{F_K(x)\}$.

But $F_{n+K}(x) = F_K(x) + n \frac{d F_K(x)}{dK} + \frac{n^2}{\underline{2}} \frac{d^2 F_K(x)}{dK^2} + \&c$

and $F_n\{F_K(x)\} = F_K(x) + n f_1\{F_K(x)\} + n^2 f_2\{F_K(x)\} + \&c$,

Equating the coeff^{ts}. of n we have $\frac{d F_K(x)}{dK} = f_1\{F_K(x)\}$

Let $F_K(x) = y$ and $F_K(y) = z$, then we have

$\frac{dy}{dK} = f_1(y)$. $\therefore \frac{dz}{dK} = f_1(y) \frac{dz}{dy}$.

$\therefore \frac{d F_K(y)}{dK} = f_1(y) \frac{d F_K(y)}{dy}$ or $\frac{d F_n(x)}{dn} = f_1(x) \frac{d F_n(x)}{dx}$.

Equating the coeff^{ts} of n^{x+1} we have $n f_n(x) = f_1(x) f_{n-1}'(x)$.

Cor 1. If $f_n(x) = \left(\frac{x}{1}\right)^n \{\psi_1(n)\, x - \psi_2(n)\, x^2 + \psi_3(n)\, x^3 - \&c\}$, then

i. $n \psi_n(n) = n \psi_1(n-1) \psi_n(1) + (n+1) \psi_2(n-1) \psi_n(1) +$

$(n+2) \psi_3(n-1) \psi_{n-2}(1) + (n+3) \psi_4(n-1) \psi_{n-3}(1) + \&c$.

ii. $\phi_n(2n) = n^n \left\{\frac{\psi_1(n-1)}{n} - \frac{\psi_2(n-2)}{n^2} + \frac{\psi_3(n-3)}{n^3} - \&c\right\}$

Sol. $n f_n(x) = f_1(x) f_{n-1}'(x)$ equate the coeff^{ts} of like powers of x. $\phi_n(n)$ is the coeff^{t} of x^n in $F_n(x)$ by I expansion

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$$\frac{1}{4n^2} + \frac{1}{n^2+(n+1)^2} + \frac{1}{n^2+(n+2)^2} + \frac{1}{n^2+(n+3)^2} + \text{&c}$$

$$= \frac{\pi}{4n} + \frac{1}{8\pi n^3} - \frac{\pi}{n} \cdot \frac{1}{e^{4\pi n} - 2e^{2\pi n}\cos 2\pi n + 1}$$

$$+ \frac{1}{4n}\left\{ \frac{1}{e^{2\pi}-1} \cdot \frac{1}{1^4+4n^4} + \frac{2}{e^{4\pi}-1} \cdot \frac{1}{2^4+4n^4} \right.$$

$$\left. + \frac{3}{e^{6\pi}-1} \cdot \frac{1}{3^4+4n^4} + \text{&c} \right\}$$

$$\frac{1}{2n^2} + \frac{1}{n^2+1^2} + \frac{1}{n^2+2^2} + \text{&c. ad inf.}$$

$$= \frac{\pi}{2n} + \frac{\pi}{n} \cdot \frac{1}{e^{2\pi n}-1}$$

$$l\left\{ \frac{1}{2(l^2+n^2)} + \frac{1}{l^2+(n+1)^2} + \frac{1}{l^2+(n+2)^2} + \text{&c} \right\}$$

$$= \tan^{-1}\frac{l}{n} + \frac{B_2}{2} \cdot \frac{\sin\left(2\tan^{-1}\frac{l}{n}\right)}{l^2+n^2} - \frac{B_4}{4} \cdot \frac{\sin\left(4\tan^{-1}\frac{l}{n}\right)}{(l^2+n^2)^2}$$

$$+ \frac{B_6}{6} \cdot \frac{\sin\left(6\tan^{-1}\frac{l}{n}\right)}{(l^2+n^2)^3} + \text{&c}$$

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: Again find the coeff^{t} of x^n by II expansion.
Equate the two results.

Cor 2. $(n+1)\,\psi_n(1) = \frac{1}{2}\psi_{n+1}(1) + \frac{B_2}{2}\psi_n^{(1)} - \frac{B_4}{2^3}\psi_{n-1}^{(3)} + \frac{B_6}{2^5}\psi_{n-5}^{(5)} - \&c.$

Sol. Equate the coeff^{ts} of x^n in IV 2 Cor 3.

4. $f_1(x) = (1+x)\,f_1\{\log_e(1+x)\}$.

Sol. In IV 1 write $\log_e(1+x)$ for x; then $F_{n-1}(x) = \log_e(1+x)$
$+ n f_1\{\log_e(1+x)\} + n^2 f_2\{\log_e(1+x)\} + \&c$

$\therefore e^{F_{n-1}(x)} = (1+x)\,e^{n f_1\{\log_e(1+x)\} + n^2 f_2\{\log_e(1+x)\} + \&c}$

But $e^{F_{n-1}(x)} = 1 + F_n(x) = 1 + x + n f_1(x) + n^2 f_2(x) + \&c$

$\therefore (1+x) + n(1+x) f_1\{\log_e(1+x)\} + \&c = (1+x) + n f_1(x) + \&c$

$\therefore$ Equating the coeff^{ts} of n $f_1(x) = (1+x) f_1\{\log_e(1+x)\}$.

5. i The sum of the coeff^{ts} of the odd terms + the sum of the coeff^{ts} of the even terms $\Big\}$ in $\phi_n(n) = \frac{1}{n}$.

ii. The sum of the coeff^{ts} of the odd terms − the sum of the coeff^{ts} of the even terms $\Big\}$ in $\phi_n(n) = \frac{1}{1n}$.

Sol. $F_1(x) = e^x - 1$ and $F_{-1}(x) = \log_e(1+x)$ Equate the coeff^{ts}.

iii. $f_2(x) = \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{48} - \frac{x^5}{180} + \frac{11x^6}{8640} - \frac{x^7}{6720} + \&c$

iv. $\phi_1(2n) = 1 \qquad \phi_2(2n) = n(n-\frac{1}{6})$
$\phi_2(2n) = n \qquad \phi_4(2n) = n(n-\frac{1}{6})(n-\frac{1}{4})$.

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$$(1+x)^n = 1 + nx(1+x)^{\frac{n-2}{2}} + \frac{n(n^2-1)}{4\lfloor 2} x^2(1+x)^{\frac{n-4}{2}} +$$
$$+ \frac{n(n^2-1)(n^2-3^2)}{4^2\lfloor 4} x^3(1+x)^{\frac{n-6}{2}} + \&c.$$

$$\left(\frac{1+\sqrt{1+4x}}{2}\right)^n = 1 + nx(1+x)^{\frac{n-2}{2}} + \frac{n(n-5)(n-7)}{4\lfloor 3} x^3(1+x)^{\frac{n-9}{2}}$$
$$+ \frac{n(n-7)(n-9)(n-11)(n-13)}{4^2\lfloor 5} x^5(1+x)^{\frac{n-15}{2}}$$

$$\frac{1+(1+x)^n}{2} = (1+x)^{\frac{n}{2}} + \frac{n^2}{4\lfloor 2} x^2(1+x)^{\frac{n-4}{2}}$$
$$+ \frac{n^2(n^2-0^2)}{4^2\lfloor 4} x^4(1+x)^{\frac{n-8}{2}} + \&c.$$

$$\frac{1}{2} + \frac{1}{2}\left(\frac{1+\sqrt{1+4x}}{2}\right)^n = (1+x)^{\frac{n}{2}} + \frac{n(n-4)}{4\lfloor 2} x^2(1+x)^{\frac{n-6}{2}}$$
$$+ \frac{n(n-6)(n-8)(n-10)}{4^2\lfloor 4} x^4(1+x)^{\frac{n-12}{2}} + \&c.$$

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$\phi_1(2n) = 1$

$\phi_2(2n) = n$

$\phi_3(2n) = n^2 - \frac{n}{6}$

$\phi_4(2n) = n^3 - \frac{5n^2}{12} + \frac{n}{24}$

$\phi_5(2n) = n^4 - \frac{13n^3}{18} + \frac{n^2}{6} - \frac{n}{90}$

$\phi_6(2n) = n^5 - \frac{77n^4}{72} + \frac{89n^3}{216} - \frac{91n^2}{1440} + \frac{11n}{4320}$

$\phi_7(2n) = n^6 - \frac{29n^5}{20} + \frac{175n^4}{216} - \frac{149n^3}{720} + \frac{91n^2}{4320} - \frac{n}{3360}$

6. $\psi_1(n) = 1$

$\psi_2(n-1) = \frac{n}{6}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right)$

$\psi_3(n-2) = \frac{n(n-1)}{72}\left\{\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right)^2 - \left(\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right) - \frac{1}{2} + \frac{1}{2}\right\}$

Cor. If $\frac{x}{1-ax} = y$ and $(1-ax) = z$, then

$F_{2a}(x) = y + \frac{y^2}{6}\log_e z + \frac{y^3}{72}\left\{(\log_e z)^2 + (1-\log_e z)^2 - 2\right\} + \&c.$

Sol. Apply the above results in V 1.

Ex. Shew that $f_1(x)\, f_1''(x) = f_1(x) - f_2(x) + 3B_2 f_3(x) - 5B_4 f_5(x) + \&c$

Sol. From V 2 Cor 3. we have $f_1'(x) = x - \frac{1}{2} f_1(x) + B_2 f_2(x)$
$- B_4 f_4(x) + B_6 f_6(x) - \&c$. Differentiating both sides and
multiplying the results by $f_1(x)$ we have

$f_1(x)\, f_1''(x) = f_1(x) - \frac{1}{2} f_1(x) f_1'(x) + B_2 f_1(x) f_1'(x) - B_4 f_1(x) f_4'(x) + \&c$

$= f_1(x) - \frac{1}{2} f_1(x) f_1'(x) + 3B_2 f_3(x) - 5B_4 f_5(x) + \&c$ by V 3.

7. i. $\Sigma 1 + \Sigma\frac{1}{2} + \Sigma\frac{1}{3} + \cdots + \Sigma\frac{1}{x} = (x+1)\Sigma\frac{1}{x} - x.$

ii. $(\Sigma 1)^2 + (\Sigma\frac{1}{2})^2 + (\Sigma\frac{1}{3})^2 + \cdots + (\Sigma\frac{1}{x})^2 = (x+1)(\Sigma\frac{1}{x})^2 - (2x+1)\Sigma\frac{1}{x} + 2x.$

iii. $(\Sigma 1)^3 + (\Sigma\frac{1}{2})^3 + (\Sigma\frac{1}{3})^3 + \cdots + (\Sigma\frac{1}{x})^3 = (x+1)(\Sigma\frac{1}{x})^3 - 3(x+1)(\Sigma\frac{1}{x})^2$
$+ 3(2x+1)\Sigma\frac{1}{x} - 6x + \Sigma\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{x^2}\right).$

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$$c + \log n + \frac{n}{1\cdot 2} + \frac{n^2}{2\cdot 2} + \frac{n^3}{3\cdot 2} + \&c$$

$$= e^x\left(\frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \cdots + \frac{\lfloor n-1}{x^n}\theta\right)$$

where $\theta = \frac{1}{3} + \frac{4}{135x} + \frac{8}{27\cdot 105x^2} - \&c$

If $\int \frac{1-e^{-x}}{x}\,dx = c + \log x$

then $h(e^{nx}-1) + \frac{h^2}{2}(e^{nx}-1-nx) + \frac{h^3}{3}(e^{nx}-1-nx-\frac{n^2x^2}{2}) + \&c$

$$= \frac{1}{x} - \frac{1}{2x^2} + \left(\frac{1}{x^3} - \frac{1}{x^4} + \&c\right)$$

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CHAPTER VII

1. If $f(x+h) - f(x) = h\phi'(x)$ then

$$f(x) = \phi(x) - \frac{h}{2}\phi'(x) + \frac{B_1}{\lfloor 2}h^2\phi''(x) - \frac{B_2}{\lfloor 4}h^4\phi'''(x) + \frac{B_3}{\lfloor 6}h^6\phi^{v}(x) - \&c$$

2. If $f(x+h) + f(x) = h\phi'(x)$ then

$$f(x) = \frac{h}{2}\phi'(x) - (2^2-1)B_1\frac{h^2}{\lfloor 2}\phi''(x) + (2^4-1)B_4\frac{h^4}{\lfloor 4}\phi^{iv}(x) - \&c$$

Sol. If we write e^x for $\phi(x)$, we see that the coeffts in R.H.S of VII 1. are the coeffts in the expansion of $\frac{h}{e^h-1}$.

Again, if we write e^x for $\phi(x)$ in VII 2 we see the coeffts in VII 2 are the coeffts in the expansion of $\frac{h}{e^h+1}$ or

$$\frac{h}{e^h-1} - \frac{2h}{e^{2h}-1}.$$

3. Let $F_n(x) = \phi(x) - \frac{n-1}{n+1}\{\phi(x+h)+\phi(x-h)\} + \frac{(n-1)(n-3)}{(n+1)(n+3)}\{\phi(x+2h)+\phi(x-2h)\} - \frac{(n-1)(n-3)(n-5)}{(n+1)(n+3)(n+5)} \times \{\phi(x+3h)+\phi(x-3h)\} + \&c$, then

i. If $f(x+h) - f(x-h) = 2h\phi'(x)$, then

$$f(x) = \frac{F_1(x)}{1} + \frac{F_3(x)}{3} + \frac{F_5(x)}{5} + \frac{F_7(x)}{7} + \&c$$

ii. If $f(x+h) + f(x-h) = 2\phi(x)$, then

$$f(x) = F_1(x) + \frac{1}{\lfloor 2}F_3(x) + \frac{1.3}{\lfloor 4}F_5(x) + \frac{1.3.5}{\lfloor 6}F_7(x) + \&c$$

4. If $f(x+h) + Kf(x) = \phi(x)$,

let $f(x) = \frac{\phi(x)\psi_0(K)}{K+1} - \frac{h}{\lfloor 1}\cdot\frac{\phi'(x)\psi_1(K)}{(K+1)^2} + \frac{h^2}{\lfloor 2}\frac{\phi''(x)\psi_2(K)}{(K+1)^3} - \&c$

then $\frac{1}{e^x+K} = \frac{\psi_0(K)}{K+1} - \frac{x}{\lfloor 1}\frac{\psi_1(K)}{(K+1)^2} + \frac{x^2}{\lfloor 2}\frac{\psi_2(K)}{(K+1)^3} - \&c$

Sol. Let $\phi(x) = e^x$, then $f(x) = \frac{e^x}{e^h+K}$.

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$$\int \phi(x) e^{-nx} dx = -e^{-nx}\left\{\frac{\phi(x)}{n} + \frac{\phi'(x)}{n^2} + \frac{\phi''(x)}{n^3} + \&c\right\}$$

$$\int \phi(x) \cos nx\, dx = \sin nx\left\{\frac{\phi(x)}{n} - \frac{\phi''(x)}{n^3} + \&c\right\}$$
$$+ \cos nx\left\{\frac{\phi'(x)}{n^2} - \frac{\phi'''(x)}{n^4} + \&c\right\}$$

$$\int_x^\infty e^{-x^2} \cos 2nx\, dx$$
$$= +e^{-x^2}\left\{\frac{\cos(2nx+\theta)}{2r} - \frac{1.\cos(2nx+3\theta)}{2^2 r^3} + \frac{1.3\cos(2nx+5\theta)}{2^3 r^5}\right.$$
$$\left. - \frac{1.3.5\cos(2nx+7\theta)}{2^4 r^7} + \&c\right\}$$

where $\tan\theta = \frac{n}{x}$ & $r = \sqrt{n^2+x^2}$

$$\int_0^\infty e^{-x^2}\left\{e^{2nx}\phi(x) + e^{-2nx}\phi(-x)\right\} dx$$
$$= \sqrt{\pi}\, e^{n^2}\left\{\phi(n) + \frac{\phi''(n)}{4} + \frac{\phi^{IV}(n)}{4.8} + \frac{\phi^{VI}(n)}{4.8.12} + \&c\right\}$$
$$= \int_0^\infty e^{n^2-x^2}\left\{\phi(n+x) + \phi(n-x)\right\}$$

$$\int_0^\infty e^{-\frac{1}{2}x^2}\left\{A_1 - \frac{x^2}{12}A_3 + \frac{x^4}{12}A_5 - \&c\right\} dx$$
$$= \sqrt{\frac{\pi}{2}}\left\{A_0 - \frac{2}{12}A_1 + \frac{2^2}{12}A_2 - \frac{2^3}{12}A_3 + \&c\right\}$$

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Cor. 1. $1^n - 2^n K + 3^n K^2 - 4^n K^3 + 5^n K^4 - \&c = \frac{\psi_n(K)}{(K+1)^{n+1}}$

Sol. $\frac{1}{e^x + K} = e^{-x} - K e^{-2x} + K^2 e^{-3x} - K^3 e^{-4x} + \&c$

$= \frac{\psi_0(K)}{K+1} - \frac{x}{\lfloor 1} \cdot \frac{\psi_1(K)}{(K+1)^2} + \frac{x^2}{\lfloor 2} \cdot \frac{\psi_2(K)}{(K+1)^3} - \&c$ by VII 4.

Equate the coeff^{ts}. of x^n.

Cor 2. $\psi_0(K) - n \frac{\psi_1(K)}{K+1} + \frac{n(n-1)}{\lfloor 2} \frac{\psi_2(K)}{(K+1)^2} - \ldots + (-1)^n \frac{\psi_n(K)}{(K+1)^n}$

$= (-1)^{n+1} \frac{K \psi_n(K)}{(K+1)^n}$.

Sol. Multiply both sides in VII 4 by $e^x + K$, then we see that the coeff^{t}. of $x^n = 0$.

Cor 3. If $\psi_n(K) = F_1(n) - F_2(n) K + F_3(n) K^2 - F_4(n) K^3 + \&c$

$\ldots\ldots + (-1)^{n+1} F_n(K) K^{n-1}$ then i. $F_{n-2}(n) = F_{n+1}(n)$.

ii. $F_n(n-1) + n F_{n-1}(n-1) + \frac{n(n+1)}{\lfloor 2} F_{n-2}(n-1) + \ldots + \frac{\lfloor n+n-1}{\lfloor n \lfloor n-1} F_0(n-1)$

$= n^n$ Sol. Equate the coeff^{ts}. of K^{n-1} in VII 4. Cor 1.

Cor 4. $F_n(n-1) = n^{n-1} - n(n-1)^{n-1} + \frac{n(n-1)}{\lfloor 2} (n-2)^{n-1} - \&c$ to $n+1$ terms

Sol. Multiply both sides in VII 4 Cor 1. by $(1+K)^{n+1}$ and then equate the coeff^{ts}. of K^{n-1}

Cor 5. $\psi_n(-1) = \lfloor n$, $\psi_n(0) = 2^{n+1}(2^{n+1} - 1) \frac{B_{n+1}}{n+1} \sin \frac{n\pi}{2}$. $\psi_0(K) = 1$.

$\psi_1(K) = 1$

$\psi_2(K) = 1 - K$

$\psi_3(K) = 1 - 4K + K^2$

$\psi_4(K) = 1 - 11K + 11K^2 - K^3$

$\psi_5(K) = 1 - 26K + 66K^2 - 26K^3 + K^4$

$\psi_6(K) = 1 - 57K + 302K^2 - 302K^3 + 57K^4 - K^5$

$\psi_7(K) = 1 - 120K + 1191K^2 - 2416K^3 + 1191K^4 - 120K^5 + K^6$

Write under each term the sum of the product of its coeff^{t}. and the no. of terms from the left and the product of the coeff^{ts}. of the preceding term and its no. of terms from above.

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i. $B_{2n} = I_{2n} + (-1)^n (F_{2n} - 1)$

where I_{2n} is ~~the nearest~~ integer to B_{2n} and F_{2n} is the sum of the reciprocals of prime nos. next to the factors of $2n$ including unity and the number itself.

ii. The numerator of B_n is divisible by the greatest odd multiple of prime to $(2^n - 1)$.

$I_0 = I_2 = I_4 = I_6 = I_8 = I_{10} = I_{12} = 0$.

$I_{14} = 1$, $I_{16} = 7$, $I_{18} = 55$, $I_{20} = 529$.

$I_{22} = 6192$, $I_{24} = 86580$, $I_{26} = 1425517$

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Cor 6. $\psi_n(x-1)$ is the integral part of

$$\frac{x^{n+1}}{1-x}\left\{\frac{\lfloor n}{\left(\log_e \frac{1}{1-x}\right)^{n+1}} - \frac{B_{n+1}}{n+1}\sin\frac{\pi n}{2}\right\}$$

Sol. $e^{-x} + e^{-2x} + e^{-3x} + \&c = \frac{1}{e^x - 1} = \frac{1}{x} - \&c$

Differentiating n times we have

$$1^n e^{-x} + 2^n e^{-2x} + 3^n e^{-3x} + \&c = \frac{\lfloor n}{x^{n+1}} \pm \&c$$

Writing $\log_e \frac{1}{1-x}$ for x we have

$$\frac{1^n}{1-x} + \frac{2^n}{(1-x)^2} + \frac{3^n}{(1-x)^3} + \&c = \frac{\lfloor n}{\left(\log_e \frac{1}{1-x}\right)^{n+1}} \pm \&c$$

Applying VI L. Cor 1. we at once get the result.

Ex. 1. Shew that $f(x)$ is the term independent of n in

$$\frac{\phi(x) + \frac{1}{n}\phi'(x) + \frac{1}{n^2}\phi''(x) + \frac{1}{n^3}\phi'''(x) + \&c}{e^{nh} + K}$$

2. If $n > r$, $F_r(n-1)$ is the coeff^t of $\frac{x^{n-1}}{\lfloor n-1}$ in $e^{x(r-n)}(e^x - 1)^n$.

3. $\frac{\cos x + K}{1 + 2K\cos x + K^2} = \frac{\psi_0(K)}{K+1} - \frac{x^2}{\lfloor 2}\frac{\psi_1(K)}{(K+1)^3} + \frac{x^4}{\lfloor 4}\frac{\psi_2(K)}{(K+1)^5} - \&c$

4. $\frac{\sin x}{1 + 2K\cos x + K^2} = \frac{x}{\lfloor 1}\cdot\frac{\psi_1(K)}{(K+1)^2} - \frac{x^3}{\lfloor 3}\cdot\frac{\psi_3(K)}{(K+1)^4} + \frac{x^5}{\lfloor 5}\cdot\frac{\psi_5(K)}{(K+1)^6} - \&c$

5. If n is even shew that $\psi_n(K)$ is divisible by $(1-K)$

6. If $1^n(S_2 - 1) - 2^n(S_3 - 1) + 3^n(S_4 - 1) - \&c = \cos\pi n - \sin\frac{\pi n}{2}\cdot\frac{B_{n+1}}{n+1}(2^{n+1} - 1) + A_1 S_2 - A_2 S_3 + A_3 S_4 - \&c$ where $S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c$, then

i. $A_n + {}^n\!/_{1}\, A_{n-1} + \frac{n(n-1)}{\lfloor 2} A_{n-2} + \frac{n(n-1)(n-2)}{\lfloor 3} A_{n-3} + \&c + A_0 = n^n$.

ii. $A_n = n^n - n(n-1)^n + \frac{n(n-1)}{\lfloor 2}(n-2)^n - \&c$

iii. $\frac{A_n}{\lfloor n}$ is the coeff^t of x^n in $(e^x - 1)^n$.

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$$\frac{3392780147 + 6960\left\{\frac{1^{27}x}{1-x} + \frac{2^{27}x^2}{1-x^2} + \frac{3^{27}x^3}{1-x^3} + \&c\right\}}{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \&c\right)}$$

$$= 489693897\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \&c\right)\right\}^6$$

$$+ 2507636250\left\{1 + 240\left(\frac{1^3x}{1-x} + \&c\right)\right\}^3\left\{1 - 504\left(\frac{1^5x}{1-x} + \&c\right)\right\}^2$$

$$+ 395450000\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \&c\right)\right\}^4,$$

$$B_{38} = \frac{2929\,993913\,841559}{6}$$

$$\frac{1723168255201 - 171864\left\{\frac{1^{29}x}{1-x} + \frac{2^{29}x^2}{1-x^2} + \frac{3^{29}x^3}{1-x^3} + \&c\right\}}{1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \&c\right)}.$$

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iv. $\psi_n(k-1) = A_n - A_{n-1}k + A_{n-2}k^2 -$ &c to n terms.

7. Shew that

i. $\frac{1^5}{2} + \frac{2^5}{2^2} + \frac{3^5}{2^3} + \frac{4^5}{2^4} +$ &c $= 1082.$

ii. $\frac{1^5}{3} + \frac{2^5}{3^2} + \frac{3^5}{3^3} + \frac{4^5}{3^4} +$ &c $= 68\frac{1}{4}.$

5. $\frac{x}{e^x - 1} = 1 - \frac{x}{2} + B_2\frac{x^2}{\lfloor 2} - B_4\frac{x^4}{\lfloor 4} + B_6\frac{x^6}{\lfloor 6} -$ &c

Sol. Write e^x for $\phi(x)$ in VI!.

Cor 1. $\frac{x}{e^x+1} = \frac{x}{2} - B_2\frac{x^2}{\lfloor 2}(2^2-1) + B_4\frac{x^4}{\lfloor 4}(2^4-1) - \frac{B_6}{\lfloor 6}x^6(2^6-1) +$ &c

Sol. $\frac{x}{e^x+1} = \frac{x}{e^x-1} - \frac{2x}{e^{2x}-1}.$

Cor 2. $\log_e \frac{x}{e^x-1} = -\frac{x}{2} - B_2\frac{x^2}{2\lfloor 2} + B_4\frac{x^4}{4\lfloor 4} - B_6\frac{x^6}{6\lfloor 6} +$ &c

Sol. $\log_e(e^x-1) = \int \frac{e^x}{e^x-1}\,dx.$

Cor 3. $\log_e \frac{2}{e^x+1} = -\frac{x}{2} - B_2\frac{x^2}{2\lfloor 2}(2^2-1) + B_4\frac{x^4}{4\lfloor 4}(2^4-1) -$ &c

Sol. $\log_e(e^x+1) = \log_e(e^{2x}-1) - \log_e(e^x-1)$

Ex. If P, Q, R, S &c are so small that $\frac{1}{710}$ of the Sum of their Cubes may be neglected Shew that

1. If $e^P + e^Q + e^R = 2 + e^{P+Q+R}$ then

$$\frac{1}{P} + \frac{1}{Q} + \frac{1}{R} + \frac{1}{2} + \frac{P+Q+R}{12} = 0$$

2. If $e^{P+Q+R+S} = \frac{e^P + e^Q + e^R + e^S - 2}{e^{-P} + e^{-Q} + e^{-R} + e^{-S} - 2}$, then

$$\frac{1}{P} + \frac{1}{Q} + \frac{1}{R} + \frac{1}{S} + \frac{P+Q+R+S}{12} = 0$$

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$$B_{22} = \frac{11(57183 + 205500)}{138}$$

$$B_{24} = \frac{236364091}{2730} = \frac{19.1617^2 + 10.4200^2 + 34.550^2}{2730}$$

$$B_{26} = \frac{8553103}{6} = \frac{13(392931 + 265000)}{6}$$

$$236364091 + 131040\left(\frac{1^{23}x}{1-x} + \frac{2^{23}x^2}{1-x^2} + \&\right)$$

$$= 49679091\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \&\right)\right\}^6$$

$$+ 176400000\left\{1 + 240\left(\frac{1^3x}{1-x} + \&\right)\right\}^3\left\{1 - 504\left(\frac{1^5x}{1-x} + \&\right)\right\}^2$$

$$+ 10285000\left\{1 - 504\left(\frac{1^5x}{1-x} + \frac{4^5x^2}{1-x^2} + \&\right)\right\}^4.$$

$$B_{28} = \frac{2374946\,1029}{870} = \frac{7}{870}\left(19.23.11^2.21^3 + 2.5^2.5^2.4549 + 55.10^4.719\right)$$

$$B_{30} = \frac{8615841276005}{14322}$$

$$B_{32} = \frac{1,709321041217}{510}$$

$$B_{34} = \frac{2577687,858367}{6}$$

$$B_{36} = \frac{26,315271,553053,477373}{1919190}$$

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3. If $2e^{P+Q+R+S+T} =$

$$\frac{(e^P+e^Q+e^R+e^S+e^T-2)^2-(e^{2P}+e^{2Q}+e^{2R}+e^{2S}+e^{2T}-2)}{e^{-P}+e^{-Q}+e^{-R}+e^{-S}+e^{-T}-2}, \text{ then}$$

$$\frac{1}{P}+\frac{1}{Q}+\frac{1}{R}+\frac{1}{S}+\frac{1}{T}+\frac{P+Q+R+S+T}{\lfloor 2}=\frac{1}{2}.$$

6. $x\cot x = 1 - B_2\frac{(2x)^2}{\lfloor 2} - B_4\frac{(2x)^4}{\lfloor 4} - B_6\frac{(2x)^6}{\lfloor 6} - B_8\frac{(2x)^8}{\lfloor 8}$ &c

Sol. Write xi for x in VI S:

N.B. From the nature of the coeffts we see that $B_1 = -1$

Cor. $(2n+1)B_{2n} = 2B_2B_{2n-2}\frac{2n(2n-1)}{\lfloor 2} + 2B_4B_{2n-4}\frac{2n(2n-1)(2n-2)}{\lfloor 4} \times (2n-3) + $ &c the last term being $2B_{n-1}B_{n+1}\frac{\lfloor 2n}{\lfloor n-1 \lfloor n+1}$ or $(B_n)^2\frac{\lfloor 2n}{(\lfloor n)^2}$ according as n is odd or even.

Sol. $\cot^2 x = -\left(1+\frac{d\cot x}{dx}\right)$ Equate the coeffts of x^{2n-2}

$B_0 = -1$, $B_2 = \frac{1}{6}$, $B_4 = \frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = \frac{1}{30}$, $B_{10} = \frac{5}{66}$, $B_{12} = \frac{691}{2730}$

$B_{14} = \frac{7}{6}$, $B_{16} = \frac{3617}{510}$, $B_{18} = \frac{43867}{798}$, $B_{20} = \frac{174611}{330}$, $B_{22} = \frac{854513}{138}$

E.g. 1. $x\operatorname{cosec} x = 1 + B_2\frac{x^2(2^2-1)}{\lfloor 2} + B_4\frac{x^4(2^4-1)}{\lfloor 4} +$ &c

Sol. $\operatorname{cosec} x = \cot\frac{x}{2} - \cot x$.

2. $x\tan x = B_2\frac{2^2-1}{\lfloor 2}(2x)^2 + B_4\frac{2^4-1}{\lfloor 4}(2x)^4 +$ &c

Sol. $\tan x = \cot x - 2\cot 2x$

3. $\log_e(x\operatorname{cosec} x) = B_2\frac{(2x)^2}{2\lfloor 2} + B_4\frac{(2x)^4}{4\lfloor 4} + B_6\frac{(2x)^6}{6\lfloor 6} +$ &c

Sol. $\log_e \sin x = \int \cot x\,dx$.

$\log_e \sec x = B_2\frac{2^2}{2\lfloor 2}\frac{1}{2}(2x)^2 + B_4\frac{2^4}{4\lfloor 4}(2x)^4 + \ldots$

or $\log_e \sec x = \int \tan x\,dx$

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If $\alpha, \beta, \gamma, \delta$ &c are the roots of the eqn $f(x) = 0$

then $f(x) = f(0)\left(1 - \frac{x}{\alpha}\right)\left(1 - \frac{x}{\beta}\right)\left(1 - \frac{x}{\gamma}\right)$ &c.

$$\pi x \left\{1 + \left(\frac{x}{n+1}\right)^{-1}\right\}\left\{1 + \left(\frac{x}{n+2}\right)^{-1}\right\}\left\{1 + \left(\frac{x}{n+3}\right)^{-2}\right\} \text{ &c}$$

$$= \left(\frac{\lfloor n}{n^n}\right)^2 \sqrt{\frac{\cosh 2\pi x - \cos 2\pi x}{2}}^{*} e^{-\frac{S_2}{x^2} + \frac{S_4}{2x^4} - \frac{S_6}{3x^6} + \text{&c}}$$

where $S_p = 1^p + 2^p + 3^p + \cdots n^p$.

$$\pi (x^2 + n^2)^{n+\frac{1}{2}} \left\{1 + \left(\frac{x}{n+1}\right)^{-1}\right\}\left\{1 + \left(\frac{x}{n+2}\right)^{-1}\right\}\left\{1 + \left(\frac{x}{n+3}\right)^{-1}\right\} \text{ &c}$$

$$= \left(\lfloor n\right)^2 \sqrt{\frac{\cosh 2\pi x - \cos 2\pi x}{2}}^{*} e^{2n - 2x\tan^{-1}\frac{x}{n} - \frac{B_2 S_2}{x} - \frac{B_4 S_4}{2x^3}}$$

$$- \frac{B_6 S_6}{6x^5} - \text{&c} \quad \text{where } S_p = \frac{n^p}{x} - \frac{p(p+1)}{\lfloor 3} \cdot \left(\frac{n}{x}\right)^3 + \frac{p(p+1)(p+2)(p+3)}{\lfloor 5} \left(\frac{n}{x}\right)^5 - \text{&c}$$

$$2\pi (n^2 + x^2)^{n-\frac{1}{2}} \left\{1 + \left(\frac{x}{n}\right)^{-2}\right\}\left\{1 + \left(\frac{x}{n+1}\right)^{-2}\right\}\left\{1 + \left(\frac{x}{n+1}\right)^{-2}\right\}$$

$$= \left(\lfloor n-1\right)^2 e^{2n + 2x\theta - \frac{2B_2\cos\theta}{1.2n} + \frac{2B_4\cos 3\theta}{3.4n^3} - \text{&c}}$$

$$\times \left(1 - e^{-2\pi x} E.\right)$$

where $n^2 = n^2 + x^2$, $\tan\theta = \frac{x}{n}$ & E the error is less than 1 & $= \cos 2\pi n$ when x is

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7. $\sec x = E_1 + \frac{x^2}{\lfloor 2} E_3 + \frac{x^4}{\lfloor 4} E_5 + \frac{x^6}{\lfloor 6} E_7 + \&c$

Cor. $\frac{B_{2n}}{\lfloor 2n} 2^{2n}(2^{2n}-1) = 2 E_1 E_{2n-1} + 2 E_3 E_{2n-3} \frac{(2n-2)(2n-3)}{\lfloor 2} + \&c$

the last term being $2 E_{n-1} E_{n+1} \frac{\lfloor 2n-2}{\lfloor n-2 \lfloor n}$ or $(E_n)^2 \frac{\lfloor 2n-2}{(\lfloor n-1)^2}$ according as n is even or odd.

Sol. $\frac{d \tan x}{dx} = \sec^2 x$. Equate the coeffts. of x^{2n-2}.

$E_1 = 1$, $E_3 = 1$, $E_5 = 5$, $E_7 = 61$, $E_9 = 1385$, $E_{11} = 50521$,

$E_{13} = 2702765$, $E_{15} = 199360981$, &c.

8. i. $\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right) \cdots$ &c

Sol. The roots of the equation $\frac{\sin x}{x} = 0$ are $\pm\pi, \pm 2\pi$, &c and $\frac{\sin x}{x} = 1$ when $x = 0$.

ii. In a similar manner

$\cos x = \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{3^2\pi^2}\right)\left(1 - \frac{4x^2}{5^2\pi^2}\right)\left(1 - \frac{4x^2}{7^2\pi^2}\right)$ &c

Cor 1. $\frac{e^x - e^{-x}}{2x} = \left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{2^2\pi^2}\right)\left(1 + \frac{x^2}{3^2\pi^2}\right)$ &c

2. $\frac{e^x + e^{-x}}{2} = \left(1 + \frac{4x^2}{\pi^2}\right)\left(1 + \frac{4x^2}{3^2\pi^2}\right)\left(1 + \frac{4x^2}{5^2\pi^2}\right)$ &c

3. $\cos x + \sin x = \left(1 + \frac{4x}{\pi}\right)\left(1 - \frac{4x}{3\pi}\right)\left(1 + \frac{4x}{5\pi}\right)$ &c

Ex 1. $\frac{\sin(x+a)}{\sin a} = \left(1 + \frac{x}{a}\right)\left(1 - \frac{x}{\pi - a}\right)\left(1 + \frac{x}{\pi + a}\right)\left(1 - \frac{x}{2\pi - a}\right) \times$

$\left(1 + \frac{x}{2\pi + a}\right)\left(1 - \frac{x}{3\pi - a}\right)\left(1 + \frac{x}{3\pi + a}\right)$ &c

2. $\frac{\cos(x+a)}{\cos a} = \left(1 + \frac{x}{\frac{\pi}{2} + a}\right)\left(1 - \frac{x}{\frac{\pi}{2} - a}\right)\left(1 + \frac{x}{\frac{3\pi}{2} + a}\right) \times$

$\left(1 - \frac{x}{\frac{3\pi}{2} - a}\right)$ &c

3. $1 + \frac{\sin x}{\sin a} = \left(1 + \frac{x}{a}\right)\left(1 + \frac{x}{\pi - a}\right)\left(1 - \frac{x}{\pi + a}\right)\left(1 - \frac{x}{2\pi - a}\right)$ &c

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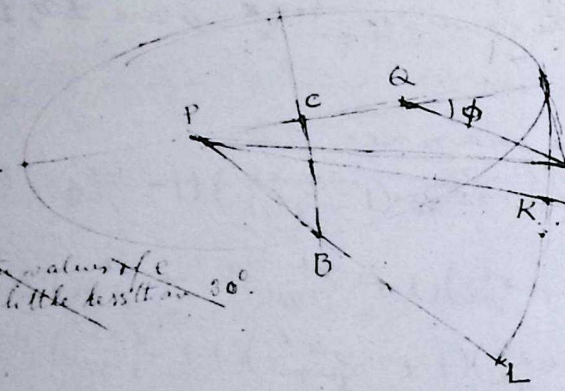
$$\int_0 \frac{\sin x}{x}\,dx = \frac{\pi}{2} - n\cos(x-\theta).$$

$$\int_0 \frac{1-\cos x}{x}\,dx = C + \log x - n\sin(x-\theta).$$

where $n^2 = \frac{11}{2^2} - \frac{13}{2^2 4} + \frac{15}{3^2 6} - \frac{17}{4^2 8} + \&c.$

$$\left.\begin{aligned} n\cos\theta &= \frac{1}{2} - \frac{12}{2^3} + \frac{14}{2^4} - \&c \\ n\sin\theta &= \frac{14}{2^2} - \frac{13}{2^3} + \frac{15}{2^4} - \&c \end{aligned}\right\}$$



$\left\{\begin{aligned} \angle MPN &= \tfrac{1}{2}\angle APN \\ CP &= CB = CQ \end{aligned}\right.$

$\frac{AL}{AK} = \frac{AB}{AN}$.

If $e = 0.021$ $\phi = 30^0$

~~For other values of e ϕ is a little less than 30^0.~~

$$\pi(a+b)\left\{1 + \frac{\sin^2\theta}{2+\cos^2\frac{\theta}{2}}\right\} \text{ where } \sin\theta = \frac{a-b}{a+b}\sin\phi.$$

when $e = 1$ $3\phi - \pi = 0^0\ 14'\ 46''$.

ϕ rapidly diminishes to 60^0.

$$\pi(a+b)\left\{1 + 4\sin^2\frac{\theta}{2}\right\} \text{ where } \sin\theta = \frac{a-b}{a+b}\sin\phi$$

when $e = 1$ $3\phi - \frac{\pi}{2} = 0^0\ 54'\ 14''$

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4. $1+\frac{\sin x}{\cos a}=\left(1+\frac{x}{\frac{\pi}{2}-a}\right)\left(1+\frac{x}{\frac{\pi}{2}+a}\right)\left(1-\frac{x}{\frac{3\pi}{2}-a}\right)\left(1+\frac{x}{\frac{3\pi}{2}+a}\right)$ &c

N.B. If we write the above results as

$\phi(x)=(1+a_1x)(1+a_2x)(1+a_3x)$ &c, then it is possible to find $(1+a_1^nx^n)(1+a_2^nx^n)(1+a_3^nx^n)$ &c

9. $\cot x=\frac{1}{x}-\frac{1}{\pi-x}+\frac{1}{\pi+x}-\frac{1}{2\pi-x}+\frac{1}{2\pi+x}-$ &c

Sol. Equate the coeff.^{s} of x in VII 8 Ex 1.

Cor 1. $\tan x=\frac{1}{\frac{\pi}{2}-x}-\frac{1}{\frac{\pi}{2}+x}+\frac{1}{\frac{3\pi}{2}-x}-\frac{1}{\frac{3\pi}{2}+x}+$ &c

2. $\operatorname{cosec} x=\frac{1}{x}+\frac{1}{\pi-x}-\frac{1}{\pi+x}-\frac{1}{2\pi-x}+$ &c

3. $\sec x=\frac{1}{\frac{\pi}{2}-x}+\frac{1}{\frac{\pi}{2}+x}-\frac{1}{\frac{3\pi}{2}-x}-\frac{1}{\frac{3\pi}{2}+x}+$ &c

10. $\tan^{-1}\frac{x}{a}-\tan^{-1}\frac{x}{\pi-a}+\tan^{-1}\frac{x}{\pi+a}-$ &c $=\tan^{-1}\left\{\frac{e^x-e^{-x}}{e^x+e^{-x}}\cot\right.$

S.-l. L.H.S $=\frac{1}{2i}\log_e\left\{\frac{1+\frac{xi}{a}}{1-\frac{xi}{a}}\cdot\frac{1-\frac{xi}{\pi-a}}{1+\frac{xi}{\pi-a}}\right.$ &c$\}$ Apply VII 8 Ex. 1

Cor 1. $\tan^{-1}\frac{x}{\frac{\pi}{2}-a}-\tan^{-1}\frac{x}{\frac{\pi}{2}+a}+\tan^{-1}\frac{x}{\frac{3\pi}{2}-a}-$ &c $=$

$\tan^{-1}\left\{\frac{e^x-e^{-x}}{e^x+e^{-x}}\tanh\right\}$.

Cor 2. $\tan^{-1}\frac{x}{1}-\tan^{-1}\frac{x}{3}+\tan^{-1}\frac{x}{5}-$ &c $=\tan^{-1}\left(\frac{e^{\frac{\pi x}{2}}-1}{e^{\frac{\pi x}{2}}+1}\right)$

Cor 3 $\tan^{-1}\frac{x}{a}+\tan^{-1}\frac{x}{\pi-a}-\tan^{-1}\frac{x}{\pi+a}-$ &c $=\tan^{-1}\left\{\frac{e^x-e^{-x}}{2}\operatorname{cosec}a\right.$

Cor 4. $\tan^{-1}\frac{x}{\frac{\pi}{2}-a}+\tan^{-1}\frac{x}{\frac{\pi}{2}+a}-\tan^{-1}\frac{x}{\frac{3\pi}{2}-a}-$ &c $=\tan^{-1}\left\{\frac{e^x+e^{-x}}{2}\sec a\right.$

Cor 5. $\tan^{-1}\frac{x}{1}+\tan^{-1}\frac{x}{3}-\tan^{-1}\frac{x}{5}-\tan^{-1}\frac{x}{7}+$ &c $=$

$\tan^{-1}\left(\frac{e^{\frac{\pi x}{2}}-1}{\sqrt{2}e^{\frac{\pi}{2}x}}\right)$.

Ex. 1. $\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}$ &c

2. $\sqrt{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{10}{9}\cdot\frac{10}{11}$ &c

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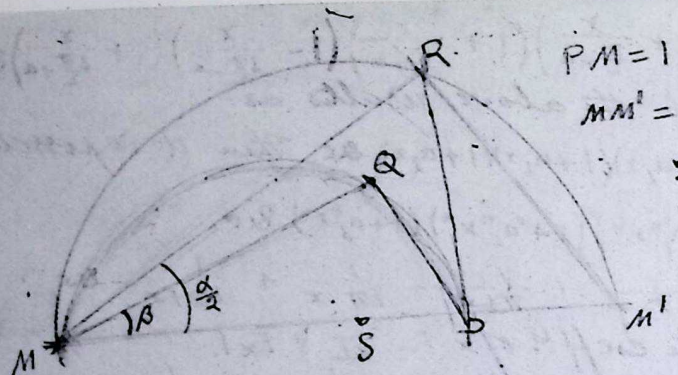
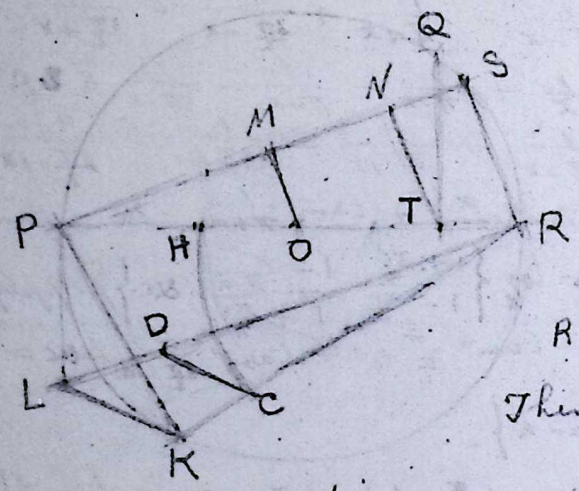


$PM = 1$; $PQ = PM'$.

$MM' = m$; $MS = M'S$.

$2PS = m \cos\alpha$.



$OH = \frac{1}{2} OP$

$RT = \frac{1}{3} OR$

$RS = TQ$

OM, TN, RS are $\perp$

$PK = PM$ & $PL = MN$

$RC = RH$ & CD is $\perp$ to LR

Thus $RD^2 = \odot$

~~From RD cut off $\frac{1}{2327\,5365}$ th of it.~~

$$\pi = \frac{355}{113}\left(1 - \frac{.0003}{3533}\right).$$

Perimeter of an ellipse $= \pi(a+b)\left\{1 + \frac{3t}{10 + \sqrt{4 - 3t}}\right\}$ very nearly

where $t = \left(\frac{a-b}{a+b}\right)^2$.

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CHAPTER VII

1. i. $\frac{1}{1^2-x^2} + \frac{1}{2^2-x^2} + \frac{1}{3^2-x^2} + \frac{1}{4^2-x^2} + \&c = \frac{1}{2x^2} - \frac{\pi}{2x}\cot\pi x.$

ii. $\frac{1}{1^2-x^2} + \frac{1}{3^2-x^2} + \frac{1}{5^2-x^2} + \frac{1}{7^2-x^2} + \&c = \frac{\pi}{4x}\tan\frac{\pi x}{2}.$

iii. $\frac{1}{1^2-x^2} - \frac{1}{2^2-x^2} + \frac{1}{3^2-x^2} - \frac{1}{4^2-x^2} + \&c = \frac{\pi}{2x}\operatorname{cosec}\pi x - \frac{1}{2x^2}.$

iv. $\frac{1}{1^2-x^2} - \frac{3}{3^2-x^2} + \frac{5}{5^2-x^2} - \&c = \frac{\pi}{4}\sec\frac{\pi x}{2}.$

Ex. 1. $\frac{1}{1^2+x^2} + \frac{1}{2^2+x^2} + \frac{1}{3^2+x^2} + \&c = \frac{\pi}{2x}\frac{e^{\pi x}+e^{-\pi x}}{e^{\pi x}-e^{-\pi x}} - \frac{1}{2x^2}.$

2. $\frac{1}{1^2+x^2} + \frac{1}{3^2+x^2} + \frac{1}{5^2+x^2} + \&c = \frac{\pi}{4x}\cdot\frac{e^{\pi x}-1}{e^{\pi x}+1}.$

3. $\frac{1}{1^2+x^2} - \frac{1}{2^2+x^2} + \frac{1}{3^2+x^2} - \&c = \frac{1}{2x^2} - \frac{\pi}{x(e^{\pi x}-e^{-\pi x})}.$

4. $\frac{1}{1^2+x^2} - \frac{3}{3^2+x^2} + \frac{5}{5^2+x^2} - \frac{7}{7^2+x^2} + \&c = \frac{\frac{\pi}{2}}{e^{\frac{\pi x}{2}}+e^{-\frac{\pi x}{2}}}$

2. i. $\frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \&c = \frac{(2\pi)^{2n}}{2\lfloor 2n}B_{2n}.$

ii. $\frac{1}{1^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \&c = \frac{(2^{2n}-1)\pi^{2n}}{2\lfloor 2n}B_{2n}.$

iii. $\frac{1}{1^{2n}} - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \&c = \frac{(2^{2n}-2)\pi^{2n}}{2\lfloor 2n}B_{2n}$

iv. $\frac{1}{1^{2n-1}} - \frac{1}{3^{2n-1}} + \frac{1}{5^{2n-1}} - \&c = \frac{\pi^{2n-1}}{2^{2n}\lfloor 2n-2}E_{2n-1}.$

N. B. From Chap. VI we know the values of B_n for even integers and E_n for odd integers, we know that B_n and E_n are infinite when $n = \infty$, that the values of B_n are all fractions though $(-1)^{n-1}B_n$ is an integer n being even and those of E_n are all integers n being odd, in both cases n being positive. Now let us try to interpret some meaning for B_n and E_n when n is positive, integral or fractional

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If $\rho = (a-b)\cos\phi = (a+b)\tan\theta$

and $\frac{\pi\rho}{\theta}$ be the perimeter of the ellipse.

then $\phi = \frac{2\sqrt{ab}}{a+b}\left\{30^\circ + 6^\circ 18' 49'' \frac{(\sqrt{a}-\sqrt{b})^2}{a+b}\right.$

$\left. = 1^\circ 10' 53'' \left(\frac{a-b}{a+b}\right)^2\right\}$

(ϕ in the figure).

~~$3\phi - 90^\circ = \frac{4ab}{(a+b)^2}\cdot\left(\frac{a-b}{a+b}\right)^2\left\{72^\circ 42' 3'' \cdot \frac{ab}{(a+b)^2}\right.$~~

~~$\left. - 2^\circ 12' 14''\right\}$~~

$\phi = 30^\circ + h(1-h)\left\{5^\circ 19\tfrac{1}{2}' - 6^\circ 3\tfrac{1}{2}' h\right\}$

where $h = \left(\frac{a-b}{a+b}\right)^2$

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Assume VII 2 to be true for all positive values of n.
To find B_n and E_n for negative values of n see chap - ter IX.

3. $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \&c = S_n = \frac{(2\pi)^n}{2\lfloor n} B_n$.

From this we can find B_n when $n > 1$.

$B_1 = \infty$, $B_3 = \frac{3}{2\pi^3} S_3$, &c

$B_{1\frac{1}{2}} = \frac{3}{4\pi\sqrt{2}} S_{1\frac{1}{2}}$ &c.

N. B. To find $\lfloor n$ for fractional values of n see chap.

4. $\frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \&c = \frac{(2^{n-1}-1)\pi^n}{\lfloor n} B_n$.

From this we can find B_n if n is not negative.

$B_0 = -1$, $B_{\frac{1}{2}} = -(1+\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \&c)$

5. $\frac{1}{1^n} - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \&c = \frac{\pi^n}{2^{n+1}\lfloor n-1} E_n$.

From this we can find E_n if n is not negative.

$E_0 = \infty$, $E_{\frac{1}{2}} = 2\sqrt{2}(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{7}} + \&c)$

$E_2 = \frac{8}{\pi^2}(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c)$.

6. $\frac{1}{(a+b)^n} + \frac{1}{(a+2b)^n} + \frac{1}{(a+3b)^n} + \&c = \frac{1}{b(n-1)a^{n-1}} - \frac{1}{2a^n}$

$+ B_2 \frac{n}{\lfloor 2} \cdot \frac{b}{a^{n+1}} - B_4 \frac{n(n+1)(n+2)}{\lfloor 4} \cdot \frac{b^3}{a^{n+3}} + \&c$

From this we can sum up the recipro-cals of pow-ers of all numbers in A.P approximately.

Sol. Let L.H.S $= \phi(a)$, then $\phi(a-b) - \phi(a) = \frac{1}{a^n}$ Apply VI 1

Cor. $S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \ldots + \frac{1}{(n-1)^n} + \frac{1}{\lfloor n^n} + \frac{1}{(n-1)n^{n-1}}$

$+ B_2 \frac{n}{\lfloor 2} \cdot \frac{1}{n^{n+1}} - B_4 \frac{n(n+1)(n+2)}{\lfloor 4} \cdot \frac{1}{n^{n+3}} + \&c$.

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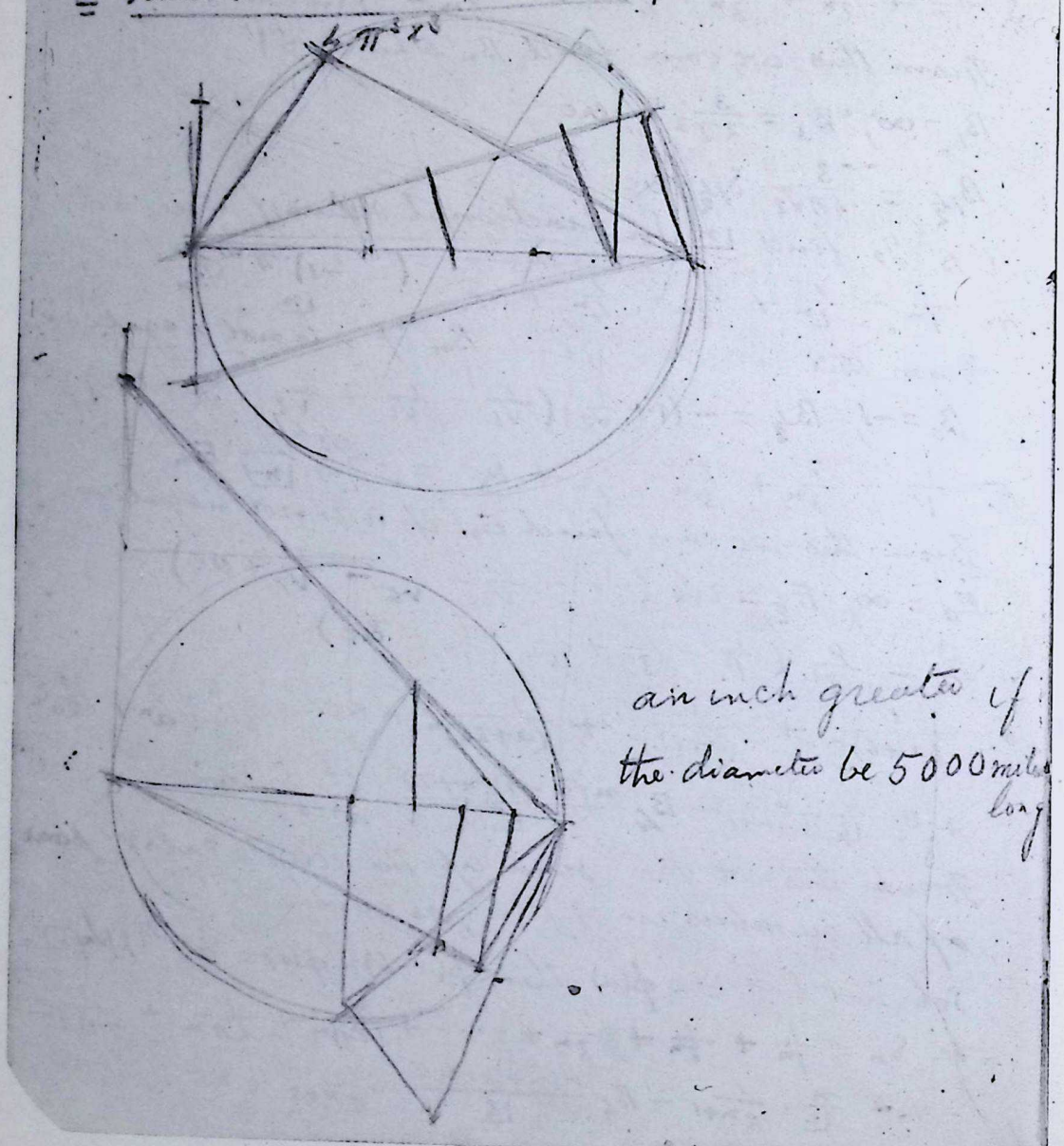
$$\left(1+\frac{x^6}{1^6}\right)\left(1+\frac{x^6}{2^6}\right)\left(1+\frac{x^6}{3^6}\right)\&$$

$$= \frac{\sinh 2\pi x - \sin 2\pi x}{4\pi^3 x^3} \cdot \frac{\cosh \pi x \cos \pi x \sqrt{3}}{}$$



an inch greater if
the diameter be 5000 miles
long

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$S_2 = 1.6449340668$ $\quad \frac{1}{B_1} = 0, \quad \frac{1}{B_2} = 6$

$S_3 = 1.2020569031$ $\quad \frac{1}{B_3} = 17.19624$

$S_4 = 1.0823232337$ $\quad \frac{1}{B_4} = 30, \quad \frac{1}{B_5} = 39.34953$

$S_5 = 1.0369277551$ $\quad \frac{1}{B_6} = 42 \quad \frac{1}{B_7} = 38.03538$

$S_6 = 1.0173430620$ $\quad \frac{1}{B_8} = 30 \quad \frac{1}{B_9} = 20.98719$

$S_7 = 1.0083492774$

$S_8 = 1.0040773561$

$S_9 = 1.0020083928$ $\quad \frac{1}{B_{10}} = 13.2.$

$S_{10} = 1.0009945781$

Ex. 1. Shew that when $n=0$ $\left(\frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \&c\right) - \frac{1}{n}$ is a finite quantity the value of which is .577 nearly. and $n S_{n+1} = 1$.

Sol. In VII 6 Cor write $n+1$ for n and 1 for r; then we have $S_{n+1} - \frac{1}{n} = \frac{1}{2} + B_2 \frac{n+1}{\lfloor 2} - \&c.$

$\therefore$ when $n=0$ $S_{n+1} - \frac{1}{n} = \frac{1}{2} + \frac{1}{12} - \frac{1}{120} + \&c$

$= .577$ nearly.

2. Shew that $\pi n B_{n+1} = 1$ when $n=0$.

Sol. We have $S_{n+1} = \frac{(2\pi)^{n+1}}{2\lfloor n+1} B_{n+1}$. Multiplying both sides by n and then writing 0 for n, we have $1 = \frac{(2\pi)^{n+1}}{2\lfloor n+1} n B_{n+1}$ when $n=0$

i.e when $n=0$ $\pi n B_{n+1} = 1$.

3. Shew that $\frac{1+E_2}{1-E_2} = \frac{\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{9^2} + \frac{1}{13^2} + \&c}{\frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{15^2} + \&c}$.

Sol. $E_2 = \frac{8}{\pi^2}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c\right)$

$= \frac{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c}{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \&c}$ Applying Componendo & divi- dendo

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1

$$(1-x^4)(1-x^3)(1-x^5)(1-x^7)(1-x^{11})(1-x^{13})(1-x^{17})\ \&c$$

$$= 1 + \frac{x^2}{1-x} + \frac{x^5}{(1-x)(1-x^4)} + \frac{x^{10}}{(1-x)(1-x^4)(1-x^3)}$$

$$+ \frac{x^{17}}{(1-x)(1-x^4)(1-x^3)(1-x^4)} + \&c$$

If $f(x) = \phi(x) + \phi(2x) + \phi(3x) + \phi(4x) + \&c$

then $\phi(x) + \phi(4x) + \phi(9x) + \phi(16x) + \&c$

$= f(x) - f(2x) - f(3x) + f(4x) - f(5x) + f(6x) - f(7x)$

$- f(8x) + f(9x) + f(10x) - f(11x) - f(12x) - \&c$

If $2, 3, 5, 7, 11$ &c are prime numbers then

$$\sqrt{\frac{a(a-1)}{4}} + \sqrt{\frac{a(a-1)}{4}} + \sqrt{\frac{a(a-1)}{4}} + \&c \text{ to } n \text{ terms} + h.$$

$$= \frac{a}{2}\left\{1 - \mu/a^n + \frac{(\mu/a^n)^2}{2(a-1)} - \frac{(\mu/a^n)^3}{2(a-1)(a^2-1)} + \frac{(\mu/a^n)^4(a+5)}{8(a-1)(a^2-1)(a^3-1)}\right.$$

$$\left. - \frac{(\mu/a^n)^5(2a^2+3a+7)}{8(a-1)(a^2-1)(a^3-1)(a^4-1)} + \&c\right\}$$

where μ is a function of a & h independent of n.

$$\frac{2h}{a} = 1 - \mu + \frac{\mu^2}{2(a-1)} - \frac{\mu^3}{2(a-1)(a^2-1)} + \&c$$

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4. Shew that

i. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \&c = \frac{\pi^2}{6}$.

ii. $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \&c = \frac{\pi^3}{32}$

iii. $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \&c = \frac{\pi^4}{90}$.

7. $\frac{1}{1-a_2} \cdot \frac{1}{1-a_3} \cdot \frac{1}{1-a_5} \cdot \frac{1}{1-a_7} \cdot \frac{1}{1-a_{11}}$. &c, where 2, 3, 5, 7, 11, 13 &c are all prime numbers

$= 1 + a_2 + a_3 + a_2 \cdot a_2 + a_5 + a_2 a_3 + a_7 + a_2 a_2 a_2 + \&c$

where the suffixes are natural numbers resolved into prime factors.

Cor 1. $\frac{2^n}{2^n-1} \cdot \frac{3^n}{3^n-1} \cdot \frac{5^n}{5^n-1} \cdot \frac{7^n}{7^n-1} \cdot \frac{11^n}{11^n-1}$ &c where 2, 3, 5, 7 &c are prime numbers $= \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c$ or S_n.

Cor 2. $\frac{2^n}{2^n+1} \cdot \frac{3^n}{3^n+1} \cdot \frac{5^n}{5^n+1}$ &c $= \frac{S_{2n}}{S_n}$

Sol. $\frac{2^n}{2^n-1} \cdot \frac{3^n}{3^n-1} \cdot \frac{5^n}{5^n-1} \&c = S_n$

$\therefore \frac{2^{2n}}{2^{2n}-1} \cdot \frac{3^{2n}}{3^{2n}-1} \cdot \frac{5^{2n}}{5^{2n}-1} \&c = S_{2n}$

$\therefore \frac{2^n}{2^n+1} \cdot \frac{3^n}{3^n+1} \cdot \frac{5^n}{5^n+1} \&c = \frac{S_{2n}}{S_n}$.

Cor 3 $\frac{2^n+1}{2^n-1} \cdot \frac{3^n+1}{3^n-1} \cdot \frac{5^n+1}{5^n-1}$ &c $= \frac{(S_n)^2}{S_{2n}}$.

Cor 4. $\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{8^n} + \frac{1}{11^n} + \frac{1}{12^n} + \&c$

where 2, 3, 5, 7, 8 are natural numbers containing an odd number of prime factors

$= \frac{(S_n)^2 - S_{2n}}{2 S_n}$.

Sol. Subtract Cor 1 from Cor 1 after applying VIII 7.

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$$\frac{1}{2\pi\sqrt{3}x^4} + \frac{1}{1^4+1^2x^2+x^4} + \frac{2}{2^4+2^2x^2+x^4} + \frac{3}{3^4+3^2x^2+x^4} + \&c$$

$$= \frac{\pi}{3x^2\sqrt{3}} \cdot \frac{\cosh \pi x\sqrt{3} + 2\cos \pi x}{\cosh \pi x\sqrt{3} - \cos \pi x}$$

$$+ 2\left\{\frac{1}{e^{\pi\sqrt{3}}+1} \cdot \frac{1}{1^4+1^2x^2+x^4} - \frac{2}{e^{2\pi\sqrt{3}}-1} \cdot \frac{1}{2^4+1^2x^2+x^4} + \&c\right\}$$

$$\frac{1}{2n^2} + \frac{1}{1^2+n+n^2} + \frac{1}{2^2+2n+n^2} + \frac{1}{3^2+3n+n^2} + \&c$$

$$+ 2n\left\{\frac{1}{e^{\pi\sqrt{3}}+1} \cdot \frac{1}{1^4+n^2+n^4} - \frac{2}{e^{2\pi\sqrt{3}}-1} \cdot \frac{1}{2^4+1^2n^2+n^4} + \&c\right\}$$

$$= \frac{1}{2\pi n^3\sqrt{3}} + \frac{2\pi}{3n\sqrt{3}} - \frac{2\pi}{n\sqrt{3}} \cdot \frac{1}{e^{2\pi n\sqrt{3}} - 2e^{\pi n\sqrt{3}}\cos \pi n + 1}$$

$$\frac{1}{6n^2} + \frac{1}{1^2+3n+3n^2} + \frac{1}{2^2+6n+3n^2} + \frac{1}{3^2+9n+3n^2} + \&c$$

$$+ 6n\left\{\frac{1}{e^{\pi\sqrt{3}}+1} \cdot \frac{1}{1^4+3n^2+9n^4} - \frac{2}{e^{2\pi\sqrt{3}}-1} \cdot \frac{1}{2^4-2^2\cdot 3n^2+9n^4} + \&c\right\}$$

$$= \frac{1}{6\pi n^3\sqrt{3}} + \frac{\pi}{3n\sqrt{3}} - \frac{2\pi}{n\sqrt{3}} \cdot \frac{1}{e^{2\pi n\sqrt{3}} - 2e^{\pi n\sqrt{3}}\cos 3\pi n + 1}$$

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Cor 5. $\frac{3^n}{3^n+1}\cdot\frac{5^n}{5^n-1}\cdot\frac{7^n}{7^n+1}\cdot\frac{11^n}{11^n+1}\cdots\cdots\frac{p^n}{p^n-\sin\frac{n\pi}{2}}\cdots$ ad inf.

where P is a prime number

$= \frac{1}{1^n} - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \&c$

8. $(1+a_2)(1+a_3)(1+a_5)(1+a_7)(1+a_{11})$ &c where 2, 3, 5, 7 &c are prime numbers

$= 1 + a_2 + a_3 + a_5 + a_2a_3 + a_7 + a_2a_5 + a_{11} + a_{13} + \&c$

where the suffixes are natural numbers resolved into prime factors no two of which are alike.

Cor 1. $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \&c = \frac{S_n}{S_{2n}}$.

Cor 2. $\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \frac{1}{19^n} + \frac{1}{23^n} + \frac{1}{29^n} + \frac{1}{30^n} + \frac{1}{31^n} + \frac{1}{37^n} + \&c = \frac{(S_n)^2 - S_{2n}}{2S_nS_{2n}}$

where 2, 3, 5, 7 &c are natural numbers containing an odd number of prime factors no two of which are alike.

Cor 3. $\frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{12^n} + \&c = \frac{S_n(S_{2n}-1)}{S_{2n}}$.

where 4, 8, 9, 12 &c are composite numbers containing at least two equal prime numbers

Ex. 1. Show that the sum of the reciprocals of all prime numbers is infinite.

Sol. From VII 7 Cor 1 we have

$\frac{2}{2-1}\cdot\frac{3}{3-1}\cdot\frac{5}{5-1}\cdot\frac{7}{7-1}\,\&c = 1 + \frac{1}{2} + \frac{1}{3} + \&c$

$\therefore \log_e\frac{2}{2-1} + \log_e\frac{3}{3-1} + \log_e\frac{5}{5-1} + \&c = \log_e(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c)$

i.e $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \&c$ + a finite quantity $= \infty$. $\therefore$ The sum of the reciprocals of all prime nos $= \infty$

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2. Shew that when $n=0$, $\log_e n + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{5^{n+1}} + \&c$ where 2, 3, 5, 7 &c are prime numbers is a finite quantity

Sol. From VII 7 Cor. we have

$$\frac{2^{n+1}}{2^{n+1}-1} \cdot \frac{3^{n+1}}{3^{n+1}-1} \cdot \frac{5^{n+1}}{5^{n+1}-1} \cdot \&c = \frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \&c$$

$$\therefore \log_e \frac{2^{n+1}}{2^{n+1}-1} + \log_e \frac{3^{n+1}}{3^{n+1}-1} + \log_e \frac{5^{n+1}}{5^{n+1}-1} + \&c = \log_e \left(\frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \&c \right)$$

$= -\log_e n$ when $n=0$

$\therefore \log_e n + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{5^{n+1}} + \&c$ is a finite quantity when $n=0$.

3. If 2, 3, 5, 7, 11 are all prime numbers shew that

i. $\frac{2^2+1}{2^2-1} \cdot \frac{3^2+1}{3^2-1} \cdot \frac{5^2+1}{5^2-1} \cdot \frac{7^2+1}{7^2-1} \cdot \frac{11^2+1}{11^2-1} \&c = \frac{5}{2}$.

ii. $\frac{2^4+1}{2^4-1} \cdot \frac{3^4+1}{3^4-1} \cdot \frac{5^4+1}{5^4-1} \cdot \frac{7^4+1}{7^4-1} \cdot \frac{11^4+1}{11^4-1} \&c = \frac{7}{6}$.

4. If 2, 3, 5, 7, 8 &c are natural numbers containing an odd no. of prime factors shew that

i. $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \&c = \frac{\pi^2}{20}$

ii. $\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{8^4} + \&c = \frac{\pi^4}{1260}$.

5. $\frac{1}{2} \sin \frac{4\pi}{2} + \frac{1}{3} \sin \frac{3\pi}{2} + \frac{1}{5} \sin \frac{5\pi}{2} + \frac{1}{7} \sin \frac{7\pi}{2} + \&c$ is a convergent series, 2, 3, 5, 7 &c being prime nos.

$$\frac{\frac{\log_e 1}{1^n} + \frac{\log_e 2}{2^n} + \frac{\log_e 3}{3^n} + \frac{\log_e 4}{4^n} + \&c}{\frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \&c} = \frac{\log_e 2}{2^n - 1} + \frac{\log_e 3}{5^n - 1} + \frac{\log_e 5}{5^n - 1} + \&c$$

where 2, 3, 5, 7, 11 are prime nos.

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$$f(1)+f(2)+f(3)+\cdots+f(h)$$
$$=\{f(1)+f(2)+f(3)+\cdots+f(n)\}$$
$$-\{f(1+h)+f(2+h)+\cdots+f(n+h)\}$$
$$+\,h\,f(n)+\frac{\Sigma h}{\lfloor 1}f'(n)+\frac{\Sigma h^2}{\lfloor 2}f''(n)+\&c$$

N.B. When n becomes infinite we may neglect $f^r(n)$ and the terms succeeding $f^r(n)$ if it is 0.

e.g. $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{h}$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$$
$$-\left(\frac{1}{1+h}+\frac{1}{2+h}+\frac{1}{3+h}+\cdots+\frac{1}{n+h}\right) \text{ when } n=\infty.$$
$$=c_0+\log_e n-\left(\frac{1}{1+h}+\frac{1}{2+h}+\frac{1}{3+h}+\cdots\frac{1}{n+h}\right) \text{ when } n=\infty.$$

$$\lfloor h=\frac{n^h}{\left(1+\frac{h}{1}\right)\left(1+\frac{h}{2}\right)\left(1+\frac{h}{3}\right)\cdots\left(1+\frac{h}{n}\right)} \text{ when } n=\infty.$$

$$f(1)+f(2)+f(3)+\cdots+f(h)$$
$$=x f(1)-x^{1+h}f(1+h)+x^2 f(2)-x^{2+h}f(2+h)+\&c$$

when x becomes unity.

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CHAPTER VIII

1. $f(1) + f(2) + f(3) + f(4) + \ldots\ldots\ldots\ldots + f(x) = \phi(x)$.

$$\phi(x) = c + \int f(x)\,dx + \tfrac{1}{2} f(x) + \frac{B_2}{\underline{2}} f'(x) - \frac{B_4}{\underline{4}} f'''(x) + \frac{B_6}{\underline{6}} f^{V}(x) - \frac{B_8}{\underline{8}} f^{VII}(x) + \&c$$

Sol. $\phi(x) - \phi(x-1) = f(x)$; apply VI 1.

N.B.1. By giving any value to x, c can be found. R.H.S is not a terminating series except in some special cases. Consequently no constant can be found in. $\frac{1}{2} f(x) + \frac{B_2}{\underline{2}} f'(x) - \frac{B_4}{\underline{4}} f'''(x) + \&c$ except in those special cases. If R.H.S be a terminating series, it must be some integral function of x; in this case there is no possibility of a constant (according to the ordinary sense) in $\phi(x)$; for, $\phi(1) = f(1) + \phi(0)$. But $\phi(1) = f(1)$; $\therefore \phi(0)$ is always 0 whether $\phi(x)$ is rational or irrational. $\therefore$ When $\phi(x)$ is a rational integral function of (x) it must be divisible by x Since $\phi(0) = 0$ and Consequently no constant but 0 can exist. Therefore let us try to give some other meaning for the constant of a series.

The constant of a series is the constant obtained by completing the remaining part by faithfully adhering to the above rule for summation of series. The constant of a series has some mysterious connection with the given infinite series and it is like the centre of gravity of a body. Mysterious because we may substitute it for the divergent infinite series. Now the constant of the series $1+1+1+1+\&c$ is $-\frac{1}{2}$; for the sum to x terms $= x = c + \int 1\,dx + \frac{1}{2}$. $\therefore c = -\frac{1}{2}$.

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$\sqrt{21}$ $\frac{1}{2}\left(\frac{3}{\sqrt{2}}\sqrt{7}\right)^2\left(\sqrt{\frac{5+\sqrt{7}}{4}}-\sqrt{\frac{1+\sqrt{7}}{4}}\right)^4\left(\sqrt{\frac{3+\sqrt{7}}{4}}-\sqrt{\frac{\sqrt{7}-1}{4}}\right)^4\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3$

$\sqrt{33}$ $\frac{1}{2}(2-\sqrt{3})^3\left(\sqrt{\frac{7+3\sqrt{3}}{4}}-\sqrt{\frac{3+3\sqrt{3}}{4}}\right)^4\left(\sqrt{\frac{5+\sqrt{3}}{4}}-\sqrt{\frac{1+\sqrt{3}}{4}}\right)^4\left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^2$

$\sqrt{45}$ $\frac{1}{2}(\sqrt{5}-2)^3\left(\sqrt{\frac{7+3\sqrt{5}}{4}}-\sqrt{\frac{3+3\sqrt{5}}{4}}\right)^4\left(\sqrt{\frac{3+\sqrt{5}}{2}}-\sqrt{\frac{\sqrt{5}+1}{2}}\right)^4\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^4$

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another way of finding the constant is as follows —
Let us take the series $1+2+3+4+5+$ &c. Let C be its constant. Then $C = 1+2+3+4+$ &c

$$\therefore 4C = \quad 4 \quad +8 \quad +\text{&c}$$

$$\therefore -3C = 1-2+3-4+\text{&c} = \frac{1}{(1+1)^2} = \frac{1}{4}.$$

$\therefore C = -\frac{1}{12}$.

N.B 2. For finding the sum to a fractional number of terms assume the sum to be true always and if there is anything difficult in finding $\phi(h)$ where h is small, take n any integer you choose, find $\phi(n+h)$ and then subtract $\{f(1+h)+f(2+h)+\cdots+f(n+h)\}$ from the result.

The sum to a negative number of terms is the sum, with the sign changed, calculated backwards from the term previous to the first to the given no. of terms with positive sign instead of negative.

Cor. $\phi(x) = -\sum_{n=0}^{n=\infty} \frac{B_n}{\lfloor n} f^{n-1}(x) \cos\frac{\pi n}{2}$.

Sol. Let $\frac{B_n}{\lfloor n}\psi(n)$ be the coeff^t. of $f^{n-1}(x)$, then we see

$\psi(0)=1,\ \psi(2)=-1,\ \psi(4)=1,\ \psi(6)=-1$ &c

$\psi(3)=0,\ \psi(5)=0,\ \psi(7)=0$ &c. $\frac{B_1}{\lfloor 1}\psi(1) = \frac{1}{2}$. But $B_1=\infty$

$\therefore \psi(1)=0$. Again by III 6 ex 2. we have $\pi(n-1)B_n = 1$ when $n=1$. $\therefore \frac{B_n\psi(n)}{\lfloor n} = \frac{\pi(n+1)B_n}{\lfloor n}\cdot\frac{\psi(n)}{\pi(n+1)} = \frac{1}{2}$ when $n=1$, i.e $\frac{\psi(n)}{\pi(n-1)} = \frac{1}{2}$ when $n=1$. $\therefore \psi(n) = -\cos\frac{\pi n}{2}$.

$$\therefore \phi(x) + \sum_{n=0}^{n=\infty}\left\{\frac{B_n}{\lfloor n} f^{n-1}(x)\cos\frac{\pi n}{2}\right\} = 0.$$

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$2 \quad (\sqrt{2}-1)^4$

$2\sqrt{3} \quad (\sqrt{3}-\sqrt{2})^4 (\sqrt{2}-1)^4$

$2\sqrt{7} \quad (\sqrt{2}-1)^8 (2\sqrt{2}-\sqrt{7})^4$

$2\sqrt{15} \quad (\sqrt{10}-3)^4 (\sqrt{6}-\sqrt{5})^4 (\sqrt{3}-\sqrt{2})^4 (\sqrt{2}-1)^4.$

$2\sqrt{2} \quad (\sqrt{3}+2\sqrt{2} \quad \sqrt{2}+2\sqrt{2})^4$

$2\sqrt{6} \quad (\sqrt{6+3\sqrt{3}} - \sqrt{5+3\sqrt{3}})^4 (\sqrt{2+\sqrt{3}} - \sqrt{1+\sqrt{3}})^4$

$2\sqrt{10} \quad (2\sqrt{2}+\sqrt{5} - 2\sqrt{3}+\sqrt{10})^4 (\sqrt{2}+\sqrt{5} - \sqrt{6+2\sqrt{10}})^4.$

If $e^{-\pi\sqrt{2p}} = F\,(\sqrt{n+1}-\sqrt{n})^2 (\sqrt{n}-\sqrt{n-1})^2$

then $e^{-2\pi\sqrt{2p}} = F \left\{ \frac{\sqrt{n}+1+\sqrt{n+1}}{\sqrt{2}} - \sqrt{(\sqrt{n}+1)(\sqrt{n}+\sqrt{n+1})} \right\}^4$

$\times \left\{ \frac{\sqrt{n}-1+\sqrt{n+1}}{\sqrt{2}} - \sqrt{(\sqrt{n}-1)(\sqrt{n}+\sqrt{n+1})} \right\}^4.$

If $e^{-\pi\sqrt{p}} = F\,\frac{1-\sqrt{1-\frac{1}{n^2}}}{2}$

then $e^{-2\pi\sqrt{p}} = F\,(\sqrt{n+1}-\sqrt{n})^4 (\sqrt{n}-\sqrt{n-1})^4,$

& $e^{-4\pi\sqrt{p}} = F \left\{ (\sqrt{n+1}+\sqrt{n})(\sqrt{2n}+1) - \sqrt{\cdots} \right\}^4$

$\times \left\{ (\sqrt{n+1}+\sqrt{n})(\sqrt{2n}-1) - \sqrt{\cdots} \right\}^4$

$= F\,(\sqrt{n+1}+\sqrt{n})^8 \left\{ \sqrt{2n}+1 - \sqrt{2\sqrt{n}(\sqrt{n+1}+\sqrt{2})} \right\}^4$

$\times \left\{ \sqrt{2n}-1 - \sqrt{2\sqrt{n}(\sqrt{n+1}-\sqrt{2})} \right\}^4.$

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2. Def. A series is said to be corrected when its constant is subtracted from it.

Theorem. – The differential coeff.t of a series is a corrected series.

i.e $\frac{d\{\phi(1)+\phi(2)+\phi(3)+\dots+\phi(x)\}}{dx} = \phi'(1)+\phi'(2)+\dots+\phi'(x)-c'$

where c' is is the constant of $\phi'(1)+\phi'(2)+\phi'(3)+\dots+\phi'(x)$.

Sol. In the differential coeff.t of $\phi(1)+\phi(2)+\dots+\phi(x)$ there can't be any constant. Therefore it should be corrected.

V.B. If $f(1)+f(2)+\dots+f(x)+\&c$ be a convergent series then its constant is the sum of the series itself.

Ex 1. $\frac{d\,\Sigma\frac{1}{x}}{dx} = \frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\&c$

Sol. $\frac{d\,\Sigma\frac{1}{x}}{dx} = -\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}-\dots-\frac{1}{x^2}-c.$

$= \frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\&c$ by VIII 2 V.B.

2. If S_1 be the constant of $\Sigma\frac{1}{x}$, then $\frac{d\lfloor x}{dx} = \lfloor x\,(\Sigma\frac{1}{x}-S_1)$.

Sol. $\frac{d\lfloor x}{dx} = \lfloor x\,\frac{d\log_e \lfloor x}{dx} = \lfloor x\,(\Sigma\frac{1}{x}-S_1)$.

3. $\int_0^x \Sigma\frac{1}{x}\,dx = \log_e \lfloor x + S_1 x$.

4. $\int_0^x (1^{13}+2^{13}+3^{13}+\dots x^{13})\,dx = \frac{1}{14}(1^{14}+2^{14}+\dots+x^{14})$

5. $\frac{d(1^{10}+2^{10}+3^{10}+\dots+x^{10})}{dx} = 10(1^9+2^9+\dots+x^9)\left(+\frac{1}{132}\right)$.

6. $\int_0^x (\sqrt{1}+\sqrt{2}+\sqrt{3}+\dots+\sqrt{x})\,dx = \frac{2}{3}(1\sqrt{1}+2\sqrt{2}+3\sqrt{3}+\dots+x\sqrt{x})$

$-\frac{x}{4\pi}\left(\frac{1}{1\sqrt{1}}+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\&c\right)$.

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If $S_n = \frac{B_n}{2n} + (-1)^{\frac{n}{2}} \left\{ \frac{1^{n-1}}{x-1} + \frac{2^{n-1}}{x^2-1} + \frac{3^{n-1}}{x^3-1} + \&c \right\}$

then $S_{n+2} \frac{(n+2)(n+3)}{2\ 1} + \frac{n(n-1)(n-2)(n-3)}{\lfloor 2} S_4 S_{n-2} (n-2)(n-13)$

$+ \frac{n \cdots (n-5)}{\lfloor 4} S_6 S_{n-4} (n-7)(n-18)$ ~~n is even~~ &c

$+ \frac{n \cdots (n-7)}{\lfloor 6} S_8 S_{n-6} \{(n-12)(n-23) - 5.1.6\}$

$+ \frac{n \cdots (n-9)}{\lfloor 8} S_{10} S_{n-8} \{(n-17)(n-28) - 5.2.7\}$

$+ \frac{n \cdots (n-11)}{\lfloor 10} S_{12} S_{n-10} \{(n-22)(n-33) - 5.3.8\} + \&c$

If the last term be a perfect by that take $\frac{1}{2}$ the term only.

n is any even no. greater than 6. $S_8 = 120 S_4^2$

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3. If $f^n(x)$ stands for the nth derivative of $f(x)$ and c_n be the constant of $\{f^n(1) + f^n(2) + \cdots + f^n(x)\}$, then

$\phi(x) = -c_1 x - c_2 \frac{x^2}{\underline{|2}} - c_3 \frac{x^3}{\underline{|3}} - c_4 \frac{x^4}{\underline{|4}} - \&c$

Sol. We know $\phi(x) = \phi(0) + \frac{x}{\underline{|1}}\phi'(0) + \frac{x^2}{\underline{|2}}\phi''(0) + \&c$

From VIII 2 we have $\phi(0) = 0$, $\phi'(0) = -c_1$, $\phi''(0) = -c_2$ &c

Ex. 1. Shew that $\log_e \underline{|x} = -s_1 x + \frac{s_2}{2} x^2 - \frac{s_3}{3} x^3 + \frac{s_4}{4} x^4 - \&c$

where s_n is the constant of $\left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c\right)$.

2. $\Sigma \frac{1}{x} = s_1 x - s_2 x^2 + s_3 x^3 - s_4 x^4 + \&c$ where $s_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c$

N. B. This is very useful in finding $\phi(x)$ for fractional values of x.

4. If c'_n be the constant of $f(\frac{1}{n}) + f(\frac{2}{n}) + f(\frac{3}{n}) + \cdots + f(\frac{x}{n})$, then

$\phi(\frac{x}{n}) + \phi(\frac{x-1}{n}) + \phi(\frac{x-2}{n}) + \cdots + \phi(\frac{x-n+1}{n}) - nc$

$= f(\frac{1}{n}) + f(\frac{2}{n}) + f(\frac{3}{n}) + \cdots + f(\frac{x}{n}) - c'_n$.

Sol. Let $\psi(x) = \phi(\frac{x}{n}) + \phi(\frac{x-1}{n}) + \cdots + \phi(\frac{x-n+1}{n})$, then

$\psi(x) - \psi(x-1) = \phi(\frac{x}{n}) - \phi(\frac{x-n}{n}) = f(\frac{x}{n})$

$\therefore \psi(x)$ and $f(\frac{x}{n}) + f(\frac{x-1}{n}) + \cdots$ differ only by some constant; hence if these be corrected they must be equal. $\psi(x)$ contains n terms each of which is of the form $\phi(\quad)$ whose constant is c. $\therefore$ The constant of $\psi(x) = nc$. And the constant of $f(\frac{1}{n}) + f(\frac{2}{n}) + f(\frac{3}{n}) + \cdots + f(\frac{x}{n})$ is c'_n by our supposition.

Cor. $\phi(-\frac{1}{n}) + \phi(-\frac{2}{n}) + \phi(-\frac{3}{n}) + \cdots + \phi(-\frac{n-1}{n}) = nc - c'_n$.

Sol. Put $x = 0$ in the above theorem.

Ex. 1 Shew that $\phi(-\frac{1}{2}) = 2c - c'_2$.

$\because c = c_n = c'_1$.

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18/ $256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta) = P$ & $\sqrt[4]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} = Q$

1,3,5,15:— $Q^3 + \frac{1}{Q^3} - \sqrt{2}\left(\frac{1}{P} + P\right) = 0.$

1,3,7,21:— $(Q^{15} + \frac{1}{Q^{15}}) - 5(Q^{13} + \frac{1}{Q^{13}}) + 5(Q^8 + \frac{1}{Q^8}) + 6(Q^4 + \frac{1}{Q^4})$
$- \left(\frac{8}{P^6} - 6 + 8P^6\right) = 0.$

1,5,7,35:— $Q^4 + \frac{1}{Q^4} - (Q^2 + \frac{1}{Q^2}) - 2(P^2 + \frac{1}{P^2}) = 0.$

1,5,11,55:— $\begin{cases} 3,1,5,15: -(Q^4 + \frac{1}{Q^4}) - 2(P^2 + \frac{1}{P^2}) + 3 = 0. \\ 5,1,3,15: -(Q^6 + \frac{1}{Q^6}) - 4(P^4 + \frac{1}{P^4}) + 10(P^2 + \frac{1}{P^2} - 1) = 0 \end{cases}$

7,1,3,21 $(Q^8 + \frac{1}{Q^8}) + 7(Q^6 + \frac{1}{Q^6}) + 14(Q^4 + \frac{1}{Q^4}) + 21(Q^2 + \frac{1}{Q^2}) + 14$
$- P(P^6 + \frac{1}{P^6}) = 0$

5,1,7,35: $(Q^6 + \frac{1}{Q^6}) + 5\sqrt{2}(Q^3 + \frac{1}{Q^3})(P + \frac{1}{P}) + 10$
$- 4(P^4 + \frac{1}{P^4}) = 0.$

5,1,11,55 $(Q^6 + \frac{1}{Q^6}) - 5(Q^4 + \frac{1}{Q^4}) + 10(Q^2 + \frac{1}{Q^2})(\frac{1}{P^2} - 1 + P^2)$
$- \left(\frac{4}{P^4} - \frac{10}{P^2} + 25 - 10P^2 + 4P^4\right) = 0.$

3,1,11,33 $Q^4 + \frac{1}{Q^4} + 3(Q^2 + \frac{1}{Q^2}) - 2(P^2 + \frac{1}{P^2}) = 0$

19/ $\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)} = P$ & $\sqrt[4]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} = Q$

1,5,7,35 $(P^4 + \frac{1}{P^4}) - (Q^6 + \frac{1}{Q^6}) + 5(Q^4 + \frac{1}{Q^4}) - 10(Q^2 + \frac{1}{Q^2}) + 15 = 0$

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3. $\phi(-\frac{1}{3}) + \phi(-\frac{2}{3}) = 3c - c_3'$

4. $\phi(-\frac{1}{4}) + \phi(-\frac{3}{4}) = 2c + c_2' - c_4'$

5. $\phi(-\frac{1}{6}) + \phi(-\frac{5}{6}) = c + c_2' + c_3' - c_6'$

6. $\phi(-\frac{1}{2}) = c + \int f(x)\,dx - \frac{2-1}{2}\frac{B_1}{\underline{1}}f'(x) + \frac{2^3-1}{2^3}\cdot\frac{B_3}{\underline{3}}f'''(x) - \&c$

$$= \sum_{n=0}^{n=\infty}\left\{\left(1 - \frac{1}{2^{n-1}}\right)\frac{B_n}{\underline{n}}f^{n-1}(x)\cos\frac{\pi n}{2}\right\}$$

5. i. $S(a_1 + a_2 + a_3 + \&c)$ means that the series is a convergent series and its sum to infinity is req^d.

ii. $C(a_1 + a_2 + a_3 + \&c)$ means that the series is a pure divergent series and its constant is req^d.

iii. $G_r(a_1 + a_2 + a_3 + \&c)$ means that the series is an oscillating series (convergent or divergent) or that the series is a pure divergent series whose sum to n terms cannot be found and consequently its constant also and that in both cases the value of the generating function is required.

N. B. Here-after the series will only be given omitting S_r or G_r and from the nature of the series we should infer whether C_r, S or G_r is req^d. Moreover if a series appears to be equal to a finite quantity we must select S, C or G_r from the nature of the series.

iv. The value of an oscillating series is only true when the series is deduced from a regular series. For example the value of $1-1+1-1+1-1\&c = \frac{1}{2}$ only when it is deduced from a regular series of the form $\{\phi(0)\}^2 - \{\phi(1)\}^2 + \{\phi(2)\}^2 - \{\phi(3)\}^2 + \&c$ Again if we take an oscillating series $a^2 - b^2 + c^2 - \&c$

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$a_1 - a_2 + a_3 - a_4 + \&c$

$= \frac{a_1}{2} + \frac{a_1 - a_2}{4} + \frac{a_1 - 2a_2 + a_3}{8} + \&c$

$a_1 - a_2 + a_3 - a_4 + \&c$

$= x a_1 - x^2 a_2 + x^3 a_3 - x^4 a_4 + \&c$

$= x \frac{a_1}{2} + x^2 \frac{a_1 - a_2}{4} + x^3 \frac{a_1 - 2a_2 + a_3}{8} + \&c$ } when x approaches unity.

48/ $\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)} = P$. 48/ $\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)} = Q$

1, 13, 5, 65 : $-(Q^6 + \frac{1}{Q^6}) - 5(Q^4 + \frac{1}{Q^4})(P + \frac{1}{P})$

$-(4P^4 + P^4) = 0.$

1, 13, 3, 39 $-(Q^4 + \frac{1}{Q^4}) - 3(Q^2 + \frac{1}{Q^2}) + 3 - (P^2 + \frac{1}{P^2}) = 0.$

~~1, 3, 11, 33~~ —

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we get the same series $1-1+1-1+\&c$ when n becomes 0 yet its value is not $\frac{1}{2}$ in this case.

V. $a_1-a_2+a_3-a_4+\&c$ is not equal to the series $(a_1-a_2)+(a_3-a_4)+(a_5-a_6)+\&c$ or to the series $a-(a_2-a_3)-(a_4-a_5)-(a_6-a_7)-\&c$. For example $1-2+3-4+5-6+\&c$ is not equal to the Series $(1-2)+(3-4)+(5-6)+\&c$ or to the Series $1-(2-3)-(4-5)-\&c$.

But $a_1-a_2+a_3-a_4+\&c = a_1-(a_2-a_3+a_4-a_5+\&c)$.

VI. $(a_1-a_2+a_3-a_4+\&c)+(b_1-b_2+b_3-b_4+\&c)$

$= (a_1+b_1)-(a_2+b_2)+(a_3+b_3)-(a_4+b_4)+\&c$

Cor. $(a_1-a_2+a_3-a_4+\&c)+(b_1-b_2+b_3-b_4+\&c)$

$= a_1+(b_1-a_2)-(b_2-a_3)+(b_3-a_4)-(b_4-a_5)+\&c$

Sol. From VIII S^{n} V we know

$a_1-a_2+a_3-a_4+\&c = a_1-(a_2-a_3+a_4-\&c)$

$\therefore (a_1-a_2+a_3-a_4+\&c)+(b_1-b_2+b_3-b_4+\&c)$

$= a_1+(b_1-b_2+b_3-b_4+\&c)+(-a_2+a_3-a_4+\&c)$

$= a_1+(b_1-a_2)-(b_2-a_3)+(b_3-a_4)-\&c$

Ex. 1. $a_1-a_2+a_3-a_4+\&c = \frac{a_1}{2}+\frac{1}{2}\{(a_1-a_2)-(a_2-a_3)+(a_3-a_4)-\&c\}$

2. $a_1-a_2+a_3-a_4+\&c = \frac{3a_1-a_2}{4}+\frac{1}{4}\{(a_1-2a_2+a_3)-(a_2-2a_3+a_4)+(a_3-2a_4+a_5)-\&c\}$.

3. $a_1-a_2+a_3-a_4+a_5-\&c = \frac{7a_1-4a_2+a_3}{8}+$

$\frac{1}{8}\{(a_1-3a_2+3a_3-a_4)-(a_2-3a_3+3a_4-a_5)+(a_3-3a_4+3a_5-a_6)-\&c$

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$\sqrt[24]{16\alpha\beta(1-\alpha)(1-\beta)} = P$ and $\sqrt[24]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} = Q$.

3. $Q^6 + \frac{1}{Q^6} = 2\left(\frac{1}{P^3} - P^3\right) = 0$

5. $Q^3 + \frac{1}{Q^3} = 2\left(\frac{1}{P^2} - P^2\right) = 0$

7. $Q^4 + \frac{1}{Q^4} = 2\left(\frac{\sqrt{2}}{P^3} - \frac{7}{2} + \sqrt{2}P^3\right) = 0.$ $= 0$

11. $Q^6 + \frac{1}{Q^6} = 2\sqrt{2}\left(\frac{2}{P^5} - \frac{11}{P^3} + 22P - 22P^3 + 11P^5 - 2P^7\right)$

12. $Q^7 + \frac{1}{Q^7} + 13\left(Q^5 + \frac{1}{Q^5}\right) + 52\left(Q^3 + \frac{1}{Q^3}\right) + 78\left(Q + \frac{1}{Q}\right)$
$- 2\left(\frac{1}{P^6} - P^6\right) = 0.$

17. $Q^9 + \frac{1}{Q^9} - 34\left(Q^6 + \frac{1}{Q^6}\right) + 17\left(Q^3 + \frac{1}{Q^3}\right)\left(\frac{4}{P^4} + 7 + 4P^4\right)$
$\therefore \left(\frac{16}{P^8} - \frac{136}{P^4} - 340 - 136P^4 + 16P^8\right) = 0.$

19. $Q^{10} + \frac{1}{Q^{10}} + 114\left(Q^6 + \frac{1}{Q^6}\right) - 190\sqrt{2}\left(Q^4 + \frac{1}{Q^4}\right)\left(\frac{1}{P^3} - P^3\right)$
$+ 19\left(Q^2 + \frac{1}{Q^2}\right)\left(\frac{8}{P^6} - 5 + 8P^6\right)$
$\therefore 4\sqrt{2}\left(\frac{4}{P^9} + \frac{19}{P^3} - 19P^3 - 4P^9\right) = 0.$

These are true even for even functions though the signs are changed in most cases.

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vii. If $\frac{a_2}{a_3}$ lies between $\frac{a_1}{a_2}$ and $\frac{a_3}{a_4}$, then
$a_1 - a_2 + a_3 - a_4 + \&c$ lies between $\frac{a_1}{a_1+a_2}$ and $a_1 - \frac{a_2}{a_2+a_3}$

E.g. $1 - 2 + 3 - 4 + \&c$ lies between $\frac{1}{3}$ and $\frac{1}{5}$ and its value is $\frac{1}{4}$. $\underline{10} - \underline{11} + \underline{12} - \underline{13} + \&c$ lies between $\frac{1}{2}$ and $\frac{2}{3}$; its value is $\frac{3}{5}$ very nearly.

But $2 - 2\frac{1}{2} + 3\frac{1}{3} - 4\frac{1}{4} + 5\frac{1}{5} - \&c$ cannot lie between $\frac{2^2}{2+2\frac{1}{2}}$ and $2 - \frac{(2\frac{1}{2})^2}{2\frac{1}{2}+3\frac{1}{3}}$ as $\frac{2\frac{1}{2}}{3\frac{1}{3}}$ is not lying between $\frac{2}{2\frac{1}{2}}$ and $\frac{3\frac{1}{3}}{4\frac{1}{4}}$.

6. $\phi_1(x) + \phi_2(x) + \phi_3(x) + \&c$ can be expanded in ascending powers of x, say $A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c$ where each of A_0, A_1, A_2 &c is a series.

Case I When A_n is a convergent series.

(i) If $A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c$ be a rapidly convergent series what is required is got.

(ii) But if it is a slowly convergent or an oscillating series, convergent or divergent (at least for some values of x)

(a) change x into a suitable function of y so that the new series in ascending powers of y may be a rapidly convergent series; Eg. let $\frac{x}{1+x} = y$, then
$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c = y + \frac{y^3}{12} + \frac{y^5}{80} + \frac{y^7}{448} + \&c$

(b) or convert it into a continued fraction.

Eg. $x - \frac{x^2}{3} + \frac{2x^3}{15} - \frac{17x^4}{315} + \&c =$

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$\sqrt[8]{\alpha\beta} = P \cdot \sqrt[8]{\frac{\beta}{\alpha}} = Q$

3. $Q^2 - Q^2 - 2\left(\frac{1}{P} - P\right) = 0$

5. $Q^3 - Q^3 + 5\left(\frac{1}{Q} - Q\right) - 4\left(\frac{1}{P^2} - P^2\right) = 0$

7. $Q^4 + Q^4 - 8\left(\frac{1}{P^3} + P^3\right) + 28\left(\frac{1}{P^2} + P^2\right) - 56\left(\frac{1}{P} + P\right) + 70 = 0$

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$\frac{x}{1}+\frac{x}{3}+\frac{x}{5}+\frac{x}{7}+\&c \quad \frac{1^0}{x}-\frac{1^1}{x^2}+\frac{1^2}{x^3}-\frac{1^3}{x^4}+\&c = \frac{1}{x+1}-\frac{1^2}{x+3}-\frac{2^2}{x+5}-\&c$

(c) or transform it into another series by applying III 7 ex. 1

Eg. $\frac{1}{x}-\frac{2}{x^2}+\frac{5}{x^3}-\frac{15}{x^4}+\frac{52}{x^5}-\&c$

$= \frac{1}{x+1}-\frac{1}{(x+1)(x+2)}+\frac{1}{(x+1)(x+2)(x+3)}-\&c$

(d) or take the reciprocal of the series and try to make it a rapidly convergent series in any way

Case II When A_n is an oscillating (convergent or divergent) or a pure divergent series.

(1) Let C_n be the constant or the value of its generating function according as it is purely divergent or not. Then the given series $= \psi(x) + C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \&c$ where $\psi(x)$ is a simple function of x.

(2) But if unfortunately $C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \&c$ be a divergent series, find some function of n (say P_n) such that the value of $P_0 + P_1 x + P_2 x^2 + \&c$ may easily found and $C_n - P_n$ may rapidly diminish as n increases. Then the given series $= \psi(x) + (C_0 - P_0) + (C_1 - P_1)x + (C_2 - P_2)x^2 + (C_3 - P_3)x^3 + \&c$

Eg. 1. $\frac{1}{x+1}-\frac{1}{x+2}+\frac{1}{x+3}-\&c = \frac{1}{x}(1-1+1-1+\&c) - \frac{1}{x^2}(1-2+3-\&c)$

$= \frac{1}{2x} - \frac{1}{4x^2} + \&c$

2. $\frac{1}{1^2 x^2}+\frac{1}{2^2 x^2}+\frac{1}{3^2 x^2}+\&c = -\frac{1}{x^2}(1+1+1+\&c)$

$-\frac{1}{x^4}(1^2+2^2+3^2+\&c) - \frac{1}{x^6}(1^4+2^4+3^4+\&c) = \frac{1}{2x^2} + \psi(x)$

$= \frac{1}{2x^2} - \frac{\pi \cot \pi x}{2x}.$

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$$\int_0^\infty \left(\frac{\sin x}{x}\right)^{2n} dx = \frac{\pi n}{(2n-1)(n+1)}\left(1+\frac{1}{2}+\cdots\cdots\frac{1}{n}\right).$$

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^{2n+1} dx = 2\pi \cdot \frac{1\cdot 2^2 \cdots (2n-1)^2}{2^2\cdot 4^2 \cdots (2n)^2}\left(1+\frac{1}{3}+\cdots\frac{1}{2n-1}\right).$$

$$1+\frac{\frac{1}{3}\cdot\frac{2}{3}}{1\cdot\frac{1}{2}}x+\frac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{5}{3}}{1\cdot 2\cdot\frac{1}{2}\cdot 1\frac{1}{2}}x^2+\&c$$

$$=\frac{\cos\left(\frac{1}{3}\sin^{-1}\sqrt{x}\right)}{\sqrt{1-x}}.$$

$$1-\frac{\frac{1}{3}\cdot\frac{2}{3}}{1\cdot\frac{1}{2}}x+\frac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{5}{3}}{1\cdot 2\cdot\frac{1}{2}\cdot 1\frac{1}{2}}x^2-\&c$$

$$=\frac{\sqrt[3]{\sqrt{1+x}+\sqrt{x}}+\sqrt[3]{\sqrt{1+x}-\sqrt{x}}}{2\sqrt{1+x}}.$$

$$1+\frac{\frac{1}{3}\cdot\frac{2}{3}}{1\cdot 1\frac{1}{2}}x+\frac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{5}{3}}{1\cdot 2\cdot 1\frac{1}{2}\cdot 2\frac{1}{2}}x^2+\&c$$

$$=\frac{3}{\sqrt{x}}\sin\left(\frac{1}{3}\sin^{-1}\sqrt{x}\right)$$

$$1-\frac{\frac{1}{3}\cdot\frac{2}{3}}{1\cdot 1\frac{1}{2}}x+\frac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{5}{3}}{1\cdot 2\cdot 1\frac{1}{2}\cdot 2\frac{1}{2}}x^2-\&c$$

$$=\frac{3}{2\sqrt{x}}\left(\sqrt[3]{\sqrt{1+x}+\sqrt{x}}-\sqrt[3]{\sqrt{1+x}-\sqrt{x}}\right)$$

$$\cos\left(2n\sin^{-1}\sqrt{x}\right),\quad \frac{\sin\left(2n\sin^{-1}\sqrt{x}\right)}{2n\sqrt{x}},\quad \frac{\cos 2n\sin^{-1}\sqrt{x}}{\sqrt{1-x}}$$

$$\frac{\cos\left(2n\sin^{-1}\sqrt{x}\right)}{2n\sqrt{x(1-x)}}.$$

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3. $\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \&c = (1+1+1+\&c)$

$- x(\log_e 1 + \log_e 2 + \log_e 3 + \&c) + \&c = -\frac{1}{2} - x\log_e\sqrt{2\pi} - \&c$

$= \frac{1}{x-1} + 1 + x + x^2 + \&c - \frac{1}{2} - x\log_e\sqrt{2\pi} - \&c$

$= \frac{1}{x-1} + \frac{1}{2} + x(1 - \log_e\sqrt{2\pi}) - \&c.$

$= \frac{1}{x-1} + \frac{1}{2} + (1 - .91894)x - \&c = \frac{1}{x-1} + .5 + .08106x - \&c$

Cor. 1. $\frac{x}{e^x+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{3x}+1} + \frac{x}{e^{4x}+1} + \&c$

$= \log_e 2 - \frac{x}{4} + (B_2)^2 \frac{x^2(2^2-1)}{2\lfloor 2} + (B_4)^2 \frac{x^4(2^4-1)}{4\lfloor 4} +$

$(B_6)^2 \frac{x^6(2^6-1)}{6\lfloor 6} + \&c.$

Sol. By chapter VII 5 cor 1. $\frac{x}{e^{nx}+1} = \frac{x}{2} - B_2\frac{nx^2(2^2-1)}{\lfloor 2} + B_4\frac{n^3x^4(2^4-1)}{\lfloor 4} - B_6\frac{n^5x^6(2^6-1)}{\lfloor 6} + \&c$

$\therefore \frac{x}{e^x+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{3x}+1} + \&c$

$= \frac{x}{2}(1+1+1+\&c) - B_2\frac{x^2(2^2-1)}{\lfloor 2}(1+2+3+4+\&c)$

$+ B_4\frac{x^4(2^4-1)}{\lfloor 4}(1^3+2^3+3^3+\&c) - \&c$

$= \psi(x) - \frac{x}{4} + (B_2)^2\frac{x^2(2^2-1)}{2\lfloor 2} + (B_4)^2\frac{x^4(2^4-1)}{4\lfloor 4} + \&c.$

Now it is reqd. to find $\psi(x)$.

The given series $= \frac{x}{e^x-1} - \frac{x}{e^{2x}-1} + \frac{x}{e^{3x}-1} - \&c$

Here the term excepting x and higher powers of x is $\log_e 2$. $\therefore \psi(x) = \log_e 2$.

Cor 2. $\frac{x}{e^x-1} + \frac{x}{e^{2x}-1} + \frac{x}{e^{3x}-1} + \frac{x}{e^{4x}-1} + \&c$

$= S_1 - \log_e x + \frac{x}{4} - (B_2)^2\frac{x^2}{2\lfloor 2} - (B_4)^2\frac{x^4}{4\lfloor 4} - (B_6)^2\frac{x^6}{6\lfloor 6} - \&c$

Sol. Proceeding as in the previous theorem we have

the series $= \psi(x) + S_1 + \frac{x}{4} - (B_2)^2\frac{x^2}{2\lfloor 2} - (B_4)^2\frac{x^4}{4\lfloor 4} - \&c$

IIB

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$$1+6\left(\frac{1}{e^{\gamma}+e^{-\gamma}+1}+\frac{1}{e^{3\gamma}+e^{-3\gamma}+1}+\&c\right)$$

$$=1+\frac{1\cdot 2}{3^2}x+\&c=Z$$

$$\frac{1^2}{e^{\gamma}+e^{-\gamma}+1}+\frac{2^2}{e^{4\gamma}+e^{-4\gamma}+1}+\&c=\frac{x}{27}Z^3$$

$$\frac{1^4}{e^{\gamma}+e^{-\gamma}+1}+\frac{2^4}{e^{2\gamma}+e^{-2\gamma}+1}+\&c=\frac{x}{27}Z^5$$

$$\frac{1^6}{e^{\gamma}+e^{-\gamma}+1}+\frac{2^6}{e^{3\gamma}+e^{-3\gamma}+1}+\&c=\frac{x}{27}\left(1+\frac{47}{3}\right)Z^7$$

$$\frac{1^8}{e^{\gamma}+e^{-\gamma}+1}+\frac{2^8}{e^{4\gamma}+e^{-4\gamma}+1}+\&c=\frac{x}{27}(1+8x)Z^9$$

$$1+2e^{-\frac{\pi y}{\sin\pi h}}+2e^{-\frac{4\pi y}{\sin\pi h}}+\&c$$

$$=\sqrt{1+\frac{h(1-h)}{u^2}x+\&c}$$

where $y=\dfrac{1+\frac{h(1-h)}{u^2}(1-x)+\&c}{1+\frac{h(1-h)}{(1+u)^2}x+\&c}$

& u can be expressed in radicals in terms of x & h.

$$\phi=\theta+3\left\{\frac{\sin 2\theta}{1+2\cos 2\gamma}+\frac{\sin 4\theta}{2(1+2\cos 4\gamma)}+\&c\right\}$$

$$\theta Z=\int_0^{\phi}\left\{1+\frac{1\cdot 2}{3^2}\cdot\frac{2}{1}\cdot x\sin^2\phi+\frac{1\cdot 2\cdot 4\cdot 5}{3^2\cdot 6^2}\cdot\frac{2\cdot 4}{1\cdot 3}x^2\sin^4\phi\,d\phi+\&c\right\}$$

$$=\int_0^{\phi}\frac{\cos\left\{\frac{1}{3}\sin^{-1}(\sqrt{x}\sin\phi)\right\}}{\sqrt{1-x\sin^2\phi}}d\phi.$$

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But we know $\frac{x}{e^x+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{3x}+1} + \&c$

$= \left\{\frac{x}{e^x-1} + \frac{x}{e^{2x}-1} + \frac{x}{e^{3x}-1} + \&c\right\} - \left\{\frac{2x}{e^{2x}-1} + \frac{2x}{e^{4x}-1} + \frac{2x}{e^{6x}-1} + \&c\right\}$

$\therefore \psi(x) - \psi(2x) = \log_e 2$. Hence $\psi(x) = -\log x$.

Ex. 1. Shew that the constant in the series

$\sqrt[10]{1} + \sqrt[10]{2} + \sqrt[10]{3} + \sqrt[10]{4} + \ldots\ldots + \sqrt[10]{x}$ is $-.4909100$.

2. Shew that $\frac{1}{2+1} + \frac{1}{2^2+1} + \frac{1}{2^3+1} + \frac{1}{2^4+1} + \&c$

$= \frac{3}{4} + \frac{1}{48}\log_e 2$ very nearly.

3. $\frac{1}{1+1\frac{1}{9}} + \frac{1}{1+(1\frac{1}{9})^2} + \frac{1}{1+(1\frac{1}{9})^3} + \frac{1}{1+(1\frac{1}{9})^4} + \&c$

$= 6.331009$

4. $\frac{1}{1\frac{1}{9}-1} + \frac{1}{(1\frac{1}{9})^2-1} + \frac{1}{(1\frac{1}{9})^3-1} + \frac{1}{(1\frac{1}{9})^4-1} + \&c$

$= 27$ nearly

7. i. $\frac{1}{x-1} + \frac{1}{x^2-1} + \frac{1}{x^3-1} + \frac{1}{x^4-1} + \frac{1}{x^5-1} + \&c$

$= \frac{1}{x}\cdot\frac{x+1}{x-1} + \frac{1}{x^2}\cdot\frac{x^2+1}{x^2-1} + \frac{1}{x^3}\cdot\frac{x^3+1}{x^3-1} + \frac{1}{x^{16}}\cdot\frac{x^4+1}{x^4-1} + \&c$.

ii. $\frac{1}{x-1} - \frac{1}{x^2-1} + \frac{1}{x^3-1} - \frac{1}{x^4-1} + \&c$. or

$\frac{1}{x+1} + \frac{1}{x^2+1} + \frac{1}{x^3+1} + \frac{1}{x^4+1} + \&c$

$= \frac{1}{x}\cdot\frac{x^2+1}{x^2-1} - \frac{1}{x^4}\cdot\frac{x^4+1}{x^4-1} + \frac{1}{x^9}\cdot\frac{x^6+1}{x^6-1} - \frac{1}{x^{16}}\cdot\frac{x^8+1}{x^8-1} + \&c$

Sol. $\frac{1}{x-1} = \frac{1}{x-1}$

$\frac{1}{x^2-1} = \frac{1}{x^2} + \frac{1}{x^2(x^2-1)}$

$\frac{1}{x^3-1} = \frac{1}{x^3} + \frac{1}{x^6} + \frac{1}{x^6(x^3-1)}$

$\frac{1}{x^4-1} = \frac{1}{x^4} + \frac{1}{x^8} + \frac{1}{x^{12}} + \frac{1}{x^{12}(x^4-1)}$

&c &c &c &c &c &c &c

$\therefore \frac{1}{x-1} + \frac{1}{x^2-1} + \frac{1}{x^3-1} + \frac{1}{x^4-1} + \&c =$

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$$\phi^2(x^{25}) + \phi^2(x^{20}) = \{\phi(x^{25}) + 2x f(x^{15}, x^{30})\}^2 + \{\phi(x^5) + 2x^4 f(x^5, x^{45})\}^2.$$

~~$\phi^2(x) - \phi^2(x^5) = 4x f(x^5) f(x^{20})$~~

$$\{\phi(x^{25}) + 2x f(x^{15}, x^{30})\}^2 + \{\phi(x^{20}) + 2x^5 f(x^5, x^{45})\}^2$$
$$= \phi^2(x) - 2\phi^2(x^5) + 3\phi^2(x^{25}).$$

$$\phi^2(x) - \phi^2(x^5) = 4x\, x(x)\, f(x^5) f(-x^{20})$$

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$\frac{1}{x}\cdot\frac{x+1}{x-1} + \frac{1}{x^4}\cdot\frac{x^2+1}{x^2-1} + \frac{1}{x^8}\cdot\frac{x^4+1}{x^4-1} + \frac{1}{x^{16}}\cdot\frac{x^8+1}{x^8-1} + \&c$

and $\frac{1}{x-1} - \frac{1}{x^2-1} + \frac{1}{x^4-1} - \frac{1}{x^8-1} + \&c =$

$\frac{1}{x}\cdot\frac{x^2+1}{x^2-1} - \frac{1}{x^4}\cdot\frac{x^4+1}{x^4-1} + \frac{1}{x^8}\cdot\frac{x^8+1}{x^8-1} - \frac{1}{x^{16}}\cdot\frac{x^{16}+1}{x^{16}-1} + \&c$.

8. $\frac{n}{1-ax} + \frac{n^2}{1-ax^2} + \frac{n^3}{1-ax^3} + \frac{n^4}{1-ax^4} + \&c$ to n terms.

$= \left\{\frac{anx}{1-ax} + \frac{(anx^2)^2}{1-ax^2} + \frac{(anx^3)^3}{1-ax^3} + \frac{(anx^4)^4}{1-ax^4} + \&c \text{ to } n \text{ terms}\right\}$

$+ \left\{\frac{n-n^{n+1}}{1-n} + a\cdot\frac{(nx)^2-(nx)^{n+1}}{1-nx} + a^2\frac{(nx^2)^3-(nx^2)^{n+1}}{1-nx^2} + \&c \text{ to } n \text{ terms}\right\}$

Sol. $\frac{n}{1-ax} = \frac{anx}{1-ax} + n.$

$\frac{n^2}{1-ax^2} = \frac{(anx^2)^2}{1-ax^2} + n^2 + an^2x^2$

$\frac{n^3}{1-ax^3} = \frac{(anx^3)^3}{1-ax^3} + n^3 + an^3x^3 + a^2n^3x^6.$

$\frac{n^4}{1-ax^4} = \frac{(anx^4)^4}{1-ax^4} + n^4 + an^4x^4 + a^2n^4x^8 + a^3n^4x^{12}.$

&c &c &c &c &c &c &c

Adding up all these to n terms we can get the result.

Cor. $\frac{n}{1-ax} + \frac{n^2}{1-ax^2} + \frac{n^3}{1-ax^3} + \frac{n^4}{1-ax^4} + \&c$

$= \left\{\frac{anx}{1-ax} + \frac{(anx^2)^2}{1-ax^2} + \frac{(anx^3)^3}{1-ax^3} + \frac{(anx^4)^4}{1-ax^4} + \&c\right\}$

$+ \left\{\frac{n}{1-n} + \frac{a(nx)^2}{1-nx} + \frac{a^2(nx^2)^3}{1-nx^2} + \frac{a^3(nx^3)^4}{1-nx^3} + \&c\right\}$

Sol. Make n infinite in the above theorem.

Ex. $\left(\frac{1}{x-1}-\frac{1}{x}\right) + \left(\frac{1}{x^2-1}-\frac{1}{x^4}\right) + \left(\frac{1}{x^3-1}-\frac{1}{x^8}\right) + \&c$

$= \frac{2}{x^{1.2}-x^{1.2}} + \frac{2}{x^{2.3}-x^{2.2}} + \frac{2}{x^{3.4}-x^{3.2}} + \frac{2}{x^{4.5}-x^{4.2}} + \&c$

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$$2^n + 6^n + 12^n + 20^n + \&c$$
$$= A_n + \frac{n}{11} A_{n+1} + \frac{n(n-1)}{12} A_{n+2} + \&c$$

where $A_n = (1^n + 2^n + 3^n + \&c)(1 + \cos \pi n)$

$$\log 2 \left\{ \frac{1}{2 \log 2} - \frac{1}{3 \log 3} + \frac{1}{4 \log 4} - \&c \right\}$$
$$+ (\log 2)^2 \left\{ \frac{1}{2 \log 2 \log 4} + \frac{1}{3 \log 3 \log 6} + \frac{1}{4 \log 4 \log 8} + \&c \right\} = 1$$

$$\int_0^\infty \frac{\cos 2n x}{\cosh \pi \sqrt{x} + \cos \pi \sqrt{x}} \, dx$$
$$= \frac{e^{-n}}{\cosh \frac{\pi}{2}} - \frac{3 e^{-9n}}{\cosh \frac{3\pi}{2}} + \frac{5 e^{-25n}}{\cosh \frac{5\pi}{2}} - \&c$$

If $\alpha\beta = \frac{\pi^3}{4}$. then

$$\frac{1}{\cosh \sqrt{\alpha} + \cos \sqrt{\alpha}} - \frac{1}{3} \cdot \frac{1}{\cosh \sqrt{3\alpha} + \cos \sqrt{3\alpha}} +$$
$$+ \frac{1}{\cosh \frac{\pi}{2} \cosh \beta} - \frac{1}{3} \cdot \frac{1}{\cosh \frac{3\pi}{2} \cosh 9\beta} + \&c$$
$$= \frac{\pi}{8}.$$

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CHAPTER IX

1. $1^n+2^n+3^n+4^n+5^n+\cdots\cdots+x^n=\phi_n(x)$.

$$\phi_n(x)=\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}+\frac{x^{n+1}}{n+1}+\frac{x^n}{2}+B_2\frac{n}{\lfloor 2}x^{n-1}-B_4\frac{n(n-1)(n-2)}{\lfloor 4}x^{n-3}$$
$$+B_6\frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 6}x^{n-5}-\&c.$$

Sol. The corrected series is found by applying VI 1.
Let C_n be the Constant. Since $\phi_n(0)=0$, $\phi_n(x)$ must be divisible by x. The coeff^t of $x^{n-n}=-\frac{\lfloor n}{\lfloor n-n\rfloor n+1}B_{n+1}\cos\frac{\pi(n+1)}{2}$.
$\therefore$ The term in dependent of $x=-\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$.
$\therefore C_n=\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$.

2. $(a+b)^n+(a+2b)^n+(a+3b)^n+\cdots+(a+xb)^n=b^n\left\{\phi_n\left(x+\frac{a}{b}\right)-\phi_n\left(\frac{a}{b}\right)\right\}$

3. $1^n-2^n+3^n-4^n+\&c=(2^{n+1}-1)\frac{B_{n+1}}{n+1}\sin\frac{\pi n}{2}$.

Sol. $(1-2^{n+1})C_n=(1-2^{n+1})(1^n+2^n+3^n+\&c)=(1^n+2^n+3^n+\&c$,
$-2^{n+1}(1^n+2^n+3^n+4^n+\&c)=1^n-2^n+3^n-4^n+\&c$

Cor. $\phi_n(-\frac{1}{2})=\left(2-\frac{1}{2^n}\right)\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$.

Sol. $\phi_n(-\frac{1}{2})=1^n-\left(\frac{1}{2}\right)^n+2^n-\left(1\frac{1}{2}\right)^n+3^n-\left(2\frac{1}{2}\right)^n+\&c$
$=-\frac{1}{2^n}(1^n-2^n+3^n-\&c)=\left(2-\frac{1}{2^n}\right)\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$.

4. $\frac{B_{1-n}}{1-n}\sin\frac{\pi n}{2}=S_n=\frac{(2\pi)^n}{2\lfloor n}B_n$.

From this we can find B_n for negative values of n.

Sol. $\frac{B_{1+n}}{1+n}\cos\frac{\pi(1+n)}{2}$ is the constant of $1^n+2^n+3^n+\&c$
$\therefore \frac{B_{1-n}}{1-n}\cos\frac{\pi(1-n)}{2}$ is that of $\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\&c=S_n$.
$\therefore \frac{B_{1-n}}{1-n}\sin\frac{\pi n}{2}=\frac{(2\pi)^n}{2\lfloor n}B_n$.

Since $C_{-n}=S_n$ S_{-n} is invariably written for C_n

Ex. 1. $B_{-2}=2S_3$; $B_{-4}=-4S_5$; $B_{-6}=6S_7$; $B_{-8}=-8S_9$. &c.

2. $\lfloor -\frac{1}{2}=\sqrt{\pi}$. Sol. $-\frac{B_{1/2}}{1\frac{1}{2}}\sin\frac{\pi}{4}=\frac{(2\pi)^{-\frac{1}{2}}}{2\lfloor -\frac{1}{2}}B_{-\frac{1}{2}}$

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If $f(x) = \Sigma \frac{P_n}{p_n - x}$ & $\phi(x) = \Sigma \frac{Q_n}{q_n - x}$

then $f(x)\phi(y) = \Sigma \left\{ \frac{P_n}{p_n - x} \phi\left(p_n \frac{y}{x}\right) + \frac{Q_n}{q_n - y} f\left(q_n \frac{x}{y}\right) \right\}$

If $\Sigma(a_n + a_{-n}) = a_0 + (a_1 + a_{-1}) + (a_2 + a_{-2}) + \&c$

then $\Sigma(a_n + a_{-n})\,\Sigma(b_n + b_{-n})$

$= \Sigma(a_n b_n + a_{-n} b_{-n}) + \{ \Sigma(a_{n+1} b_{n-1} + a_{1-n} b_{-1-n})$

$+ \Sigma(a_{-n-1} b_{-n+1} + a_{n-1} b_{n+1}) \}$

$+ \{ \Sigma(a_{n+2} b_{n-2} + a_{2-n} b_{-n-2}) + \Sigma(a_{-n-2} b_{-n+2} + a_{n-2} b_{n+2}) \}$

$+ \&c$

$+ \{ \Sigma(a_{1-n} b_n + a_{1+n} b_{-n}) + \Sigma(a_{-n} b_{n-1} + a_n b_{-n-1}) \}$

$+ \{ \Sigma(a_{2-n} b_{1+n} + a_{2+n} b_{1-n}) + \Sigma(a_{-1-n} b_{n-2} + a_{-1+n} b_{-n-1}) \}$

$+ \&c$

$\pi^2 x^2 \frac{\cos\theta x \cosh\theta x}{\sin\pi x \sinh\pi x}$

$= 1 + 4\pi x^4 \left\{ \frac{\cos\theta \cosh\theta}{(1^4 - x^4)\sinh\pi} \right.$

$\left. - \frac{2\cos 2\theta \cosh 2\theta}{(2^4 - x^4)\sinh 2\pi} + \&c \right.$

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Again $\frac{B_{-\frac{1}{2}}}{-\frac{1}{2}} \sin\frac{\pi}{4} = \frac{(2\pi)^{\frac{1}{2}}}{2\lfloor\frac{1}{2}} B_{\frac{1}{2}}$.

Multiplying the two results we have

$\frac{1}{2}\cdot\frac{4}{3} B_{-\frac{1}{2}} B_{\frac{1}{2}} = \frac{2\pi}{2\lfloor\frac{1}{2}} \cdot \frac{1}{2\lfloor-\frac{1}{2}} B_{\frac{1}{2}} B_{-\frac{1}{2}}$ or $\frac{2}{3} = \frac{2\pi}{3\lfloor\frac{1}{2}\lfloor-\frac{1}{2}}$

$\therefore \lfloor-\frac{1}{2}\lfloor-\frac{1}{2} = \pi$. $\therefore \lfloor-\frac{1}{2} = \sqrt{\pi}$.

3. In a similar manner we can prove that $\lfloor n-1 \lfloor -n = \frac{\pi}{\sin \pi n}$.

4. Shew that

$\pi\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{6}} - \frac{1}{\sqrt{6}+\sqrt{8}} + \&c\right)$

$= \frac{1}{1\sqrt{1}} + \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} + \frac{1}{7\sqrt{7}} + \&c$.

Sol. L.H.S $= \frac{\pi}{\sqrt{2}}\{1 - (\sqrt{2}-1) + (\sqrt{3}-\sqrt{2}) - (\sqrt{4}-\sqrt{3}) + \&c\}$

$= \pi\sqrt{2}\,(\sqrt{1} - \sqrt{2} + \sqrt{3} - \sqrt{4} + \sqrt{5} - \&c)$ by VIII 5 ex 1.

$= 2(2\sqrt{2}-1)\frac{B_{\frac{1}{2}}}{\lfloor\frac{1}{2}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{\pi}{\sqrt{2}}$ by IX 3.

$= \frac{8}{3}\cdot\frac{\pi}{\sqrt{2}}\left(1 - \frac{1}{2\sqrt{2}}\right) \cdot \frac{2\lfloor\frac{1}{2}}{(2\pi)^{\frac{1}{2}}}\left(\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \&c\right)$

$= \left(1 - \frac{1}{2\sqrt{2}}\right)\left(\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \&c\right) = \frac{1}{1\sqrt{1}} + \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} + \&c$.

5. $2\pi\left(\frac{1}{\sqrt[3]{1}} - \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[3]{4}} + \&c\right)$ 15 [illegible]

$= \left(\sqrt[3]{4\pi} + \sqrt[3]{2\pi}\right)\left(\frac{1}{\sqrt[3]{1}} - \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{9}} - \frac{1}{\sqrt[3]{16}} + \&c\right)$.

6. Shew that, when x becomes infinite,

i. $\sqrt{2(2x+1)} - \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{x}}\right)$

$= (\sqrt{2}+1)\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)$.

ii. $\frac{2}{3}\sqrt{(x+\frac{1}{4})(x+\frac{1}{2})(x+\frac{3}{4})} - (\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{x})$

$= \frac{1}{4\pi}\left(\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \&c\right)$.

iii. $\frac{2}{5}\sqrt{x(x+\frac{1}{4})(x+\frac{1}{2})(x+\frac{3}{4})(x+1) + \frac{5}{768}(x+\frac{1}{2})}$

$- (1\sqrt{1} + 2\sqrt{2} + 3\sqrt{3} + \cdots + x\sqrt{x})$

$= \frac{3}{16\pi^2}\left(\frac{1}{1\sqrt{1}} + \frac{1}{4\sqrt{2}} + \frac{1}{9\sqrt{3}} + \frac{1}{16\sqrt{4}} + \&c\right)$.

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$$\frac{\pi}{4}\operatorname{Sec}\frac{\pi x}{2}\operatorname{Sech}\frac{\pi y}{2}$$

$$= \frac{1\operatorname{Sech}\frac{\pi x}{2y}}{1^2+y^2} - \frac{3\operatorname{Sech}\frac{3\pi x}{2y}}{3^2+y^2} + \frac{5\operatorname{Sech}\frac{5\pi x}{2y}}{5^2+y^2} - \&c$$

$$+ \frac{1\operatorname{Sech}\frac{\pi y}{2x}}{1^2-x^2} - \frac{3\operatorname{Sech}\frac{3\pi y}{2x}}{3^2-x^2} + \frac{5\operatorname{Sech}\frac{5\pi y}{2x}}{5^2-x^2} - \&c$$

$$\frac{\pi}{4}\operatorname{Sec}\frac{\pi x}{2}\operatorname{Sech}\frac{\pi y}{2}.\ \overline{\phi(xy) - \phi(-xy)}.$$

~~$= x\left\{\frac{\operatorname{Sech}\frac{\pi y}{2x}}{1^2-x^2}\cdot\overline{\phi(y) - \phi(-y)}\right.$~~

$$= x\left\{\frac{\phi(y)-\phi(-y)}{1^2-x^2}\operatorname{Sech}\frac{\pi y}{2x} - \frac{\phi(3y)-\phi(-3y)}{3^2-x^2}\operatorname{Sech}\frac{3\pi y}{2x}\right.$$

$$\left. + \frac{\phi(5y)-\phi(-5y)}{5^2-x^2}\operatorname{Sech}\frac{5\pi y}{2x} - \&c\right\}$$

$$+ y/i\left\{\frac{\phi(xi)-\phi(-xi)}{1^2+y^2}\operatorname{Sech}\frac{\pi x}{2y} - \frac{\phi(3xi)-\phi(-3xi)}{3^2+y^2}\operatorname{Sech}\frac{3\pi x}{2y}\right.$$

$$\left. + \frac{\phi(5xi)-\phi(-5xi)}{5^2+y^2}\operatorname{Sech}\frac{5\pi x}{2y} - \&c\right\}$$

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5. The following theorem is applied in the solution of IX 1. More general theorems will be found in

If $x=0$, the value of the generating function of the series

$$x^n\phi(0) + \frac{n}{\underline{1}}x^{n-1}\phi(1) + \frac{n(n-1)}{\underline{2}}x^{n-2}\phi(2) + \&c = \phi(n).$$

Sol. The given series =

$$\phi(0)\left\{x^n + \frac{n}{\underline{1}}x^{n-1} + \frac{n(n-1)}{\underline{2}}x^{n-2} + \&c\right\}$$
$$+\phi'(0)\left\{\frac{1n}{\underline{1}}x^{n-1} + \frac{2n(n-1)}{\underline{2}}x^{n-2} + \frac{3n(n-1)(n-2)}{\underline{3}}x^{n-3} + \&c\right\}.$$
$$+\frac{1}{2}\phi''(0)\left\{\frac{1^2n}{\underline{1}}x^{n-1} + \frac{2^2n(n-1)}{\underline{2}}x^{n-2} + \frac{3^2n(n-1)(n-2)}{\underline{3}}x^{n-3} + \&c\right\}$$
$$+ \&c$$
$$= \phi(0)(1+x)^n + n\phi'(0)(1+x)^{n-1} + \frac{1}{2}\phi''(0)\left\{n(n-1)(1+x)^{n-2} + n(1+x)^{n-1}\right\}$$
$$+ \&c = \phi(0) + \frac{n}{\underline{1}}\phi'(0) + \frac{n^2}{\underline{2}}\phi''(0) + \&c \text{ when } x=0$$
$$= \phi(n).$$

Cor. when $x=0$

$$\frac{\phi(1)}{x} - \frac{\phi(2)}{x^2} + \frac{\phi(3)}{x^3} - \frac{\phi(4)}{x^4} + \&c = \phi(0).$$

Sol. Write -1 for x in the above theorem.

Eg. ~~Let~~ $\phi(n) = \frac{1}{n}\sin\frac{\pi n}{2}$ then $\phi(0) = \frac{\pi}{2}$.

$\therefore$ when $x=0$ $\quad \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \&c = \frac{\pi}{2}$.

which is same as saying $\tan^{-1}\infty = \frac{\pi}{2}$.

Ex. 1. If $x=0$ Shew that

$$\frac{\pi}{2\sqrt{x}} - \frac{1}{2}\cdot\frac{1}{x\sqrt{x}} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{x^3\sqrt{x}} - \frac{1.3.5.7.9}{2.4.6.8.10}\cdot\frac{1}{x^5\sqrt{x}} + \&c = \sqrt{2}.$$

2. If $x=0$ then $\frac{1}{x} - \frac{\underline{1}}{x^2} + \frac{\underline{2}}{x^3} - \frac{\underline{3}}{x^4} + \&c = \infty$.

N.B. Here L.H.S $= \frac{1}{x+1} - \frac{1^2}{x+3} - \frac{1^2}{x+5} - \frac{3^2}{x+7} - \&c$ when $x=0$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \&c = \infty.$$

or L.H.S $= \frac{x}{\underline{1}} + \frac{x^2}{\underline{2}}(1+\frac{1}{2}) + \frac{x^3}{\underline{3}}(1+\frac{1}{2}+\frac{1}{3}) + \&c = e^x(C + \log_e x)$

$= \infty$ when $x=0$.

3. If $x=0$ then $x^n + \frac{n}{\underline{1}}x^{n-1} + n(n-1)x^{n-2} + n(n-1)(n-2)x^{n-3} + \&c$

$= \underline{n}$ for all values of n.

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$$\pi^2 x^2 \cot \pi x \coth \pi x$$
$$= 1 - 4\pi x^4 \left\{ \frac{\coth \pi}{1^4 - x^4} + \frac{2 \coth 2\pi}{2^4 - x^4} + \frac{3 \coth 3\pi}{3^4 - x^4} + \&c \right\}$$

$$\pi^2 xy \cot \pi x \coth \pi y$$
$$= 1 + 2\pi xy \left\{ \frac{\coth \frac{\pi x}{y}}{1 + y^2} + \frac{2 \coth \frac{2\pi x}{y}}{2^2 + y^2} + \&c \right\}$$
$$- 2\pi xy \left\{ \frac{\coth \frac{\pi y}{x}}{1^2 - x^2} + \frac{2 \coth \frac{2\pi y}{x}}{2^2 - x^2} + \&c \right\}$$

$$(\pi x)^2 \cdot \frac{\cosh \pi x\sqrt{2} + \cos \pi x\sqrt{2}}{\cosh \pi x\sqrt{2} - \cos \pi x\sqrt{2}}$$

$$= 1 + 4\pi x^4 \left\{ \frac{\coth \pi}{1^4 + x^4} + \frac{2 \coth 2\pi}{2^4 + x^4} + \frac{3 \coth 3\pi}{3^4 + x^4} + \&c \right\}$$

$$\frac{\pi}{8} \sec \frac{\pi x}{2} \operatorname{Sech} \frac{\pi x}{2}$$
$$= \frac{1^3 \operatorname{Sech} \frac{\pi}{2}}{1^4 - x^4} - \frac{3^3 \operatorname{Sech} \frac{3\pi}{2}}{3^4 - x^4} + \frac{5^3 \operatorname{Sech} \frac{5\pi}{2}}{5^4 - x^4} - \&c$$

$$\frac{\pi}{4} \cdot \frac{1}{\cosh \frac{\pi x}{\sqrt{2}} + \cos \frac{\pi x}{\sqrt{2}}}$$
$$= \frac{1^3 \operatorname{Sech} \frac{\pi}{2}}{1^4 + x^4} - \frac{3^3 \operatorname{Sech} \frac{3\pi}{2}}{3^4 + x^4} + \frac{5^3 \operatorname{Sech} \frac{5\pi}{2}}{5^4 + x^4} - \&c$$

$$\int_0^\infty \frac{\cos nx}{1 + x^4} dx = \frac{\pi}{8} e^{-n} (\cos n + \sin n)$$

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4. If $x=0$. Shew that

$$\frac{1}{x} - \frac{\lfloor 2}{x^3} + \frac{\lfloor 4}{x^5} - \frac{\lfloor 6}{x^7} + \&c = \frac{\pi}{2}.$$

Sol. Write $\lfloor n-1 \sin\frac{\pi n}{2}$ for $\phi(n)$ in the above cor.

Then $\phi(0) = \lfloor n \cdot \frac{\sin\frac{\pi n}{2}}{n}$ when $n=0 = \frac{\pi}{2}$.

N.B. Thus we are able to find exact values when $x=0$ from the above theorem and cor. though the generating functions may be too difficult to find.

The generating function in ex. 4.

$$= \frac{\pi}{2}\cos x + (C_0 + \log_e x)\sin x - \left\{1\frac{x}{\lfloor 1} - (1+\tfrac{1}{2}+\tfrac{1}{3})\frac{x^3}{\lfloor 3} + \&c\right\}$$

$= \frac{\pi}{2}$ when $x=0$.

6. $(a+b)^n - (a+2b)^n + (a+3b)^n - \&c = b^n\left\{\phi_n\left(\frac{a}{2b}\right) - \phi_n\left(\frac{a-b}{2b}\right)\right\}$

Ex. (a) If $x^2+x=y$ and $x+\frac{1}{2}=a$ shew that

(1) $\phi_1(x) = \frac{y}{4}$; (2) $\phi_2(x) = a\frac{y}{3}$; (3) $\phi_3(x) = \frac{y^2}{4}$, (4) $\phi_4(x) = \frac{a}{5}y(y-\frac{1}{3})$;

(5) $\phi_5(x) = \frac{y^2}{8}(y-\frac{1}{2})$; (6) $\phi_6(x) = \frac{a}{7}y(y^2+y+\frac{1}{3})$; (7) $\phi_7(x) = \frac{y^2}{8}(y^2-\frac{4}{3}y+\frac{1}{3})$

(8) $\phi_8(x) = \frac{a}{9}y(y^3-2y^2+\frac{9}{5}y-\frac{3}{5})$; (9) $\phi_9(x) = \frac{y^2}{10}(y^3-\frac{5}{2}y^2+3y-1\frac{1}{2})$

(10) $\phi_{9'}(x) = \frac{y^2}{10}(y-1)(y^2-\frac{3}{2}y+\frac{3}{2})$; (11) $\phi_{10}(x) = \frac{a}{11}y(y-1)(y^3-\frac{7}{3}y^2+\frac{10}{3}y-\frac{5}{3})$

(12) $\phi_{11}(x) = \frac{y^2}{12}(y^4-4y^3+8\frac{1}{2}y^2-10y+5)$.

(6). 1. $y^2 = 4\phi_2(x) = 4\{\phi_1(x)\}^2$.

2. $ay^2 = 5\phi_4(x) + \phi_2(x) = 6\phi_1(x)\phi_2(x)$

3. $y^3 = 6\phi_5(x) + 2\phi_3(x) = 8\phi_1(x)\phi_3(x)$

4. $ay^3 = 7\phi_6(x) + 5\phi_4(x) = 12\phi_2(x)\phi_3(x)$

5. $y^4 = 8\phi_7(x) + 8\phi_5(x) = 16\phi_3(x)\phi_3(x)$.

6. $ay^4 = 9\phi_8(x) + 14\phi_6(x) + \phi_4(x) = 24\phi_1(x)\phi_2(x)\phi_3(x)$.

7. $y^5 = 10\phi_9(x) + 20\phi_7(x) + 2\phi_5(x)$.

8. $ay^5 = 11\phi_{10}(x) + 30\phi_8(x) + 7\phi_6(x)$.

9. $y^6 = 12\phi_{11}(x) + 40\phi_9(x) + 12\phi_7(x)$.

7. If n is a $+^{ve}$ integer then $\phi_n(x-1) + (-1)^n\phi_n(-x) = 0$.

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$$\frac{\left|\frac{x+l+n-m-1}{2}\right| \left|\frac{x+l-n-m-1}{2}\right| \left|\frac{x-l+n+m-1}{2}\right| \left|\frac{x-l-n+m-1}{2}\right|}{\left|\frac{x-l+n-m-1}{2}\right| \left|\frac{x-l-n-m-1}{2}\right| \left|\frac{x+l+n+m-1}{2}\right| \left|\frac{x+l-n+m-1}{2}\right|} = P$$

then $\frac{1-P}{1+P} = \frac{2lmx}{x^2+l^2+m^2-n^2-1} + \frac{4(x^2-1)(l^2-1)(m^2-1)}{3(x^2+l^2+m^2-n^2-5)} + \&c$

The expansion only is true.

$$\frac{1}{x^2} - \frac{2\cos\theta}{1^2-x^2} + \frac{2\cos 2\theta}{2^2-x^2} - \frac{2\cos 3\theta}{3^2-x^2} + \&c$$

$$= \frac{\pi}{x}\{\cot \pi x \cos\theta x + \sin\theta x\}$$

$$\frac{\sin\theta}{1^2-x^2} - \frac{\sin 3\theta}{3^2-x^2} + \frac{\sin 5\theta}{5^2-x^2} - \&c = \frac{\pi}{4x} \sec\frac{\pi x}{2} \sin\theta x.$$

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Sol. Let $\psi(x) = \phi_n(x-1) + (-1)^n \phi_n(-x)$, then we see that $\psi(x+1) - \psi(x) = 0$. $\therefore$ If n is a positive integer $\psi(x) = 0$.

Ex. 1. If n is odd shew that $\phi_n(x-1) = \phi_n(-x)$.

2. Shew that $\phi_n(x)$ is divisible by $x^2(x+1)^2$ or $x(x+\frac{1}{2})(x+1)$ according as n is odd or even and point out the exceptional case; in both cases n being a positive integer.

8. $\phi_n(x) = -B_n x \cos\frac{\pi x}{2} - \frac{n}{\lfloor 2} B_{n-1} x^2 \sin\frac{\pi n}{2} + \frac{n(n-1)}{\lfloor 3} B_{n-2} x^3 \cos\frac{\pi n}{2}$
$+ \frac{n(n-1)(n-2)}{\lfloor 4} B_{n-3} x^4 \sin\frac{\pi n}{2} - \&c$
$= -n x S_{1-n} - \frac{n(n-1)}{\lfloor 2} x^2 S_{2-n} - \frac{n(n-1)(n-2)}{\lfloor 3} x^3 S_{3-n} - \&c.$

Sol. Apply VIII 3.

9. $\phi_n(x) = 1^n - (1+x)^n + 2^n - (2+x)^n + 3^n - (3+x)^n + \&c.$

10. $\phi_n(x) - n^n \left\{ \phi_n\left(\frac{x}{n}\right) + \phi_n\left(\frac{x-1}{n}\right) + \phi_n\left(\frac{x-2}{n}\right) + \dots + \phi_n\left(\frac{x-n+1}{n}\right) \right\}$
$= (n^{n+1} - 1) \frac{B_{n+1}}{n+1} \sin\frac{\pi n}{2} = (1 - n^{n+1}) S_{-n}.$

Sol. Apply VIII 4.

Cor. $\phi_n(-\frac{1}{n}) + \phi_n(-\frac{2}{n}) + \phi_n(-\frac{3}{n}) + \dots + \phi_n(-\frac{n-1}{n}) = (n - n^{-n}) S_{-n}.$

11. If n is a negative integer, then $= (n^{-n} - n) \frac{B_{n+1}}{n+1} \sin\frac{\pi n}{2}.$

$\phi_n(x-1) + (-1)^n \phi_n(-x) = \{1 + (-1)^n\} S_n + \frac{(-1)^n D_{-(n+1)x} \pi \cot \pi x}{\lfloor -n-1}$

Sol. From X we have $\phi_{-1}(x-1) - \phi_{-1}(x) = -\pi \cot \pi x$

Differentiate both sides n times.

N.B. The following method is very useful in finding the derivatives of $\pi \cot \pi x$. Let $\pi \cot \pi x = y$

The coeffts of the coeffts of π^n are the same as those in the expansion of $(\tan^{-1}\frac{1}{y})^{-n}$.

Each derivative is divisible by y^2+1 so that the last term can be exactly found.

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$$\left\{ \frac{\left|\frac{\alpha+\beta+\gamma+\delta-\epsilon-1}{2}\right| \left|\frac{\alpha+\beta+\gamma-\delta+\epsilon-1}{2}\right| \left|\frac{\alpha+\beta-\gamma+\delta+\epsilon-1}{2}\right| \left|\frac{\alpha-\beta+\gamma+\delta+\epsilon-1}{2}\right|}{\left|\frac{\alpha+\beta-\gamma-\delta-\epsilon-1}{2}\right| \left|\frac{\alpha-\beta+\gamma-\delta-\epsilon-1}{2}\right| \left|\frac{\alpha-\beta-\gamma+\delta-\epsilon-1}{2}\right| \left|\frac{\alpha-\beta-\gamma-\delta+\epsilon-1}{2}\right|} \times \right\} = Q$$

$$\left\{ \frac{\left|\frac{\alpha+\beta+\gamma+\delta+\epsilon-1}{2}\right| \left|\frac{\alpha+\beta+\gamma-\delta-\epsilon-1}{2}\right| \left|\frac{\alpha+\beta-\gamma-\delta+\epsilon-1}{2}\right| \left|\frac{\alpha-\beta-\gamma+\delta+\epsilon-1}{2}\right|}{\left|\frac{\alpha-\beta+\gamma+\delta-\epsilon-1}{2}\right| \left|\frac{\alpha-\beta+\gamma-\delta+\epsilon-1}{2}\right| \left|\frac{\alpha+\beta-\gamma+\delta-\epsilon-1}{2}\right| \left|\frac{\alpha-\beta-\gamma-\delta-\epsilon-1}{2}\right|} \times \right\} = P$$

$$\# \frac{8\alpha\beta\gamma\delta\epsilon}{\left\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\epsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2-1)^2-2^2\right\}} +$$

$$\frac{84(\alpha^2-1)(\beta^2-1)(\gamma^2-1)(\delta^2-1)(\epsilon^2-1)}{3\left\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\epsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2-5)^2-6^2\right\}} + \text{etc.}$$

$$= \frac{P-Q}{P+Q}.$$ if any one of $\alpha, \beta, \gamma, \delta, \epsilon$ be an integer.

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πy
$\pi^2(y^2+1)$
$\pi^3(y^3+y)$
$\pi^4(y^4+\frac{4}{3}y^2+\frac{1}{3})$
$\pi^5(y^5+\frac{5}{3}y^3+\frac{2}{3}y)$
$\pi^6(y^6+2y^4+\frac{17}{15}y^2+\frac{2}{15})$
$\pi^7(y^7+\frac{7}{3}y^5+\frac{77}{45}y^3+\frac{17}{45}y)$
$\pi^8(y^8+\frac{8}{3}y^6+\frac{12}{5}y^4+\frac{248}{315}y^2+\frac{17}{315})$
$\pi^9(y^9+3y^7+\frac{16}{5}y^5+\frac{88}{63}y^3+\frac{62}{315}y)$
$\pi^{10}(y^{10}+\frac{10}{3}y^8+\frac{37}{9}y^6+\frac{424}{189}y^4+\frac{1382}{2835}y^2+\frac{62}{2835})$

Write under each term the coefficient obtained by dividing the sum of the products of the coeff.t and the index of that term and of the preceding term by the index of π.

Ex. (a) For all values of n shew that

1. $\phi_n(x) - 2^n\{\phi_n(\frac{x}{2}) + \phi_n(\frac{x-1}{2})\} = (1-2^{n+1})S_{-n}$.
2. $\phi_n(-\frac{1}{2}) = (2-\frac{1}{2^n})S_{-n}$
3. $\phi_n(-\frac{1}{3}) + \phi_n(-\frac{2}{3}) = (3-\frac{1}{3^n})S_{-n}$
4. $\phi_n(-\frac{1}{4}) + \phi_n(-\frac{3}{4}) = (2+\frac{1}{2^n}-\frac{1}{4^n})S_{-n}$
5. $\phi_n(-\frac{1}{8}) + \phi_n(-\frac{5}{8}) = (1+\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{8^n})S_{-n}$.

(b) If n is a positive odd integer shew that

1. $\phi_n(-\frac{1}{3}) = (3-\frac{1}{3^n})\frac{S_{-n}}{2}$
2. $\phi_n(-\frac{1}{4}) = (1+\frac{1}{2^{n+1}}-\frac{1}{2^{2n+1}})S_{-n}$
3. $\phi_n(-\frac{1}{8}) = (1+\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{7^n})\frac{S_{-n}}{2}$
4. $\phi_n(-\frac{1}{5}) + \phi_n(-\frac{2}{5}) = (5-\frac{1}{5^n})\frac{S_{-n}}{2}$.
5. $\phi_n(-\frac{1}{8}) + \phi_n(-\frac{3}{8}) = (2+\frac{1}{2^{2n+1}}-\frac{1}{2^{3n+1}})S_{-n}$
6. $\phi_n(-\frac{1}{10}) + \phi_n(-\frac{3}{10}) = (5+\frac{1}{5^n}-\frac{1}{10^n})\frac{S_{-n}}{2}$.
7. $\phi_n(-\frac{1}{12}) + \phi_n(-\frac{5}{12}) = (6+\frac{1}{6^n}-\frac{1}{12^n})\frac{S_{-n}}{2}$.

(c) Shew that IX 11. is true even for positive integral values of n and hence deduce IX 7 from IX 11.

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$$\frac{\phi(\log 1)}{1} + \frac{\phi(\log 2)}{2} + \frac{\phi(\log 3)}{3} +$$

$$= \int_0^\infty \phi(x)\,dx + C_0\phi(0) + \frac{C_1}{\underline{1}}\phi'(0) +$$

$$2n + \frac{2^n - 2^{-n}}{1.2} + \frac{3^n - 3^{-n}}{2.3} + \frac{4^n - 4^{-n}}{3.4} + \&c$$

$$= \frac{1}{n} - \pi \cot \pi n + \&c$$

$$\frac{1}{2(2^n-1)} + \frac{1}{3(3^n-1)} + \frac{1}{4(4^n-1)} +$$

$$= \frac{.7946786 - \log_e n}{n} + .2113922$$

$$- .0060680\,n - .0000028\,n^3 + \&c$$

$$\frac{1}{2\log_e 2} + \frac{1}{3\log_e 3} + \frac{1}{4\log_e 4} + \cdots + \frac{1}{n\log_e n}$$

$$= .7946786 + \log_e \log_e \left(n + \tfrac{1}{2}\right) \text{ nearly.}$$

$$\frac{1}{2^{n+1}\log_e 2} + \frac{1}{3^{n+1}\log_e 3} + \frac{1}{4^{n+1}\log_e 4} +$$

$$= .7946786 - \log_e n + .4227843\,n$$

$$- .0364079\,n^2 + .001617\,n^3$$

$$+ .000085\,n^4 - .00002\,n^5 \&c$$

$$+ .2174630$$

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(d). 1. $\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \&c = \frac{7}{8} S_3$.

2. $\frac{1}{1^3} + \frac{1}{4^3} + \frac{1}{7^3} + \frac{1}{10^3} + \&c = \frac{2\pi^3}{81\sqrt{3}} + \frac{13}{27} S_3$.

3. $\frac{1}{1^3} + \frac{1}{5^3} + \frac{1}{9^3} + \frac{1}{13^3} + \&c = \frac{\pi^3}{64} + \frac{7}{16} S_3$.

4. $\frac{1}{1^3} + \frac{1}{7^3} + \frac{1}{13^3} + \frac{1}{19^3} + \&c = \frac{\pi^3}{36\sqrt{3}} + \frac{91}{216} S_3$.

12. $2^n \left\{ \phi_n(-\frac{1}{6}) - \phi_n(-\frac{5}{6}) \right\} = (2^n + 1) \left\{ \phi_n(-\frac{1}{3}) - \phi_n(-\frac{2}{3}) \right\}$.

Sol. $\phi_n(-\frac{1}{3}) - 2^n \left\{ \phi_n(-\frac{1}{6}) + \phi_n(-\frac{2}{3}) \right\} = (2^{n+1} - 1) S_{-n}$

& $\phi_n(-\frac{2}{3}) - 2^n \left\{ \phi_n(-\frac{1}{3}) + \phi_n(-\frac{5}{6}) \right\} = (2^{n+1} - 1) S_{-n}$ } log. IX. 10.

$\therefore 2^n \left\{ \phi_n(-\frac{1}{6}) - \phi_n(-\frac{5}{6}) \right\} = (2^n + 1) \left\{ \phi_n(-\frac{1}{3}) - \phi_n(-\frac{2}{3}) \right\}$.

13. If c_n be the constant of $\frac{(\log 1)^n}{1} + \frac{(\log 2)^n}{2} + \frac{(\log 3)^n}{3} + \&c$ then

$S_{n+1} = \frac{1}{n} + c_0 - \frac{n}{1\underline{1}} c_1 + \frac{n^2}{1\underline{2}} c_2 - \frac{n^3}{1\underline{3}} c_3 + \&c$.

Sol. It has been proved in VII 6 ex. 1. that $S_{n+1} - \frac{1}{n}$ is finite when $n = 0$. The remaining part is obtained by applying VIII 6 Case II.

The above result may be written as follows.—

$\frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{4^{n+1}} + \&c$

$= \frac{1}{n} + .5772156649 + .0728158455\, n$

$- (.00485\, n^2 + .00034\, n^3) + \frac{(\frac{n}{10})^4}{1 + \frac{n}{10}} \theta$.

where θ may be taken as equivalent to $1 + \frac{2-n}{50}$.

N.B. The above result is true for all values of n.

Ex. 1. $\frac{1}{1\sqrt[10]{1}} + \frac{1}{2\sqrt[10]{2}} + \frac{1}{3\sqrt[10]{3}} + \&c = 10.5844842$.

2. $\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \&c = 2.612315$

3. $\frac{1}{1^2\sqrt{1}} + \frac{1}{2^2\sqrt{2}} + \frac{1}{3^2\sqrt{3}} + \&c = 1.341490$.

4. $B_{1/2} = .4409932$. 5. $B_{\frac{1}{2}} = -1.032627$

6. $B_{\frac{1}{3}} = -.9420745$ & $B_{-\frac{1}{3}} = -1.3841347$

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$$\int_0^\infty e^{-x}\left(1+\frac{x}{n}\right)^{n-h}dx$$

$$= 1+\left(1-\frac{h}{n}\right)+\left(1-\frac{h}{n}\right)\left(1-\frac{h+1}{n}\right)+\left(1-\frac{h}{n}\right)\left(1-\frac{h+1}{n}\right)\left(1-\frac{h+2}{n}\right)+\&c$$

$$= \frac{e^n\lfloor n-h}{2n^{n-h}}+A_0-\frac{A_1}{n}+\frac{A_2}{n^2}-\&c$$

$$A_0=\frac{2}{3}-h. \qquad A_1=\frac{4}{135}-\frac{h^2(1-h)}{3}$$

$$A_2=\frac{8}{2835}+\frac{2h(1-h)}{135}-\frac{h(1-h^2)(2-3h^2)}{45}. \quad \&c$$

$$1+\frac{\phi(h.\overline{\alpha+\delta})}{\phi(h.\overline{\beta+\gamma})}+\frac{\phi(h.\overline{\alpha+\delta})\,\phi(h.\overline{\alpha+2\delta})}{\phi(h.\overline{\beta+\gamma})\,\phi(h.\overline{\beta+2\gamma})}+\&c.$$

$$= \sqrt{\frac{\pi\,\phi(0)}{2(\gamma-\delta)h\,\phi'(0)}}+\frac{1}{3}\cdot\frac{\gamma+\delta}{\gamma-\delta}\left\{1-\frac{\phi(0)}{\phi'(0)}\cdot\frac{\phi''(0)}{\phi'(0)}\right\}+\frac{\alpha-\beta}{\gamma-\delta}.$$

if h is very small.

$$1+\frac{x^2}{2}\cdot\frac{1}{2n+1}+\frac{x^4}{2.4}\cdot\frac{1}{(2n+1)(2n+3)}+\&c$$

$$= \frac{2^{n-1}\lfloor n-\frac{1}{2}}{x^n\sqrt{\pi}}\left\{e^x\left\{1-\frac{n(n-1)}{2}\cdot\frac{1}{x}+\frac{(n+1)n(n-1)(n-2)}{2.4}\cdot\frac{1}{x^2}-\&c\right\}\right.$$

$$\left.+\theta e^{-x}\cos\pi n\right]$$

where $\theta = e^{\frac{n(n-1)}{2x+1}}$ nearly.

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7. $B_{-\frac{1}{2}} = -1.847228.$

8. Shew that $S_{1+n} + S_{1-n} = \dfrac{2S_0}{1 + \cdot 00839n^2 + \cdot 00001n^4 + \&c}$

14. $\dfrac{\phi_n(x-1) - \phi_n(-x)}{4\lfloor n} = -\cos\frac{\pi n}{2}\left\{\dfrac{\sin 2\pi x}{(2\pi)^{n+1}} + \dfrac{\sin 4\pi x}{(4\pi)^{n+1}} + \&c\right\}.$

Sol. $\phi_n(x-1) - \phi_n(-x)$
$= (1-x)^n - x^n + (2-x)^n - (1+x)^n + (3-x)^n - (2+x)^n + \&c$
Then arrange the terms in ascending powers of x and substitute $\frac{B_n}{n}\cos\frac{\pi n}{2}$ for S_{1-n}. Similarly

15. $\dfrac{\phi_n(x-1) + \phi_n(-x) - 2S_{-n}}{4\lfloor n} = \sin\frac{\pi n}{2}\left\{\dfrac{\cos 2\pi x}{(2\pi)^{n+1}} + \dfrac{\cos 4\pi x}{(4\pi)^{n+1}} + \&c\right\}$

N.B. The above two theorems are true for all values of x when n is an integer but when n is fractional they are true only when x lies between 0 and 1.

16. $\dfrac{(2\pi q)^n}{4\lfloor n-1}\left\{\phi_{n-1}\left(\frac{p}{q}-1\right) - \phi_{n-1}\left(\frac{p}{q}\right)\right\}$ when $\frac{p}{q}$ lies between 0 & 1

$= -\sin\frac{\pi n}{2}\left[\left\{S_n - \phi_{-n}\left(\frac{1}{q}-1\right)\right\}\sin\frac{2\pi p}{q} + \left\{S_n - \phi_{-n}\left(\frac{2}{q}-1\right)\right\}\sin\frac{4\pi p}{q}\right.$
$\left. + \left\{S_n - \phi_{-n}\left(\frac{3}{q}-1\right)\right\}\sin\frac{6\pi p}{q} + \cdots + \left\{S_n - \phi_{-n}\left(\frac{q-1}{q}-1\right)\right\}\sin\frac{(2q-2)\pi p}{q}\right]$

17. $\dfrac{(2\pi q)^n}{4\lfloor n-1}\left\{\phi_{n-1}\left(\frac{p}{q}-1\right) + \phi_{n-1}\left(-\frac{p}{q}\right) - 2S_{1-n}\left(1 - \frac{1}{q^n}\right)\right\}$ for the same limits

$= -\cos\frac{\pi n}{2}\left[\left\{S_n - \phi_{-n}\left(\frac{1}{q}-1\right)\right\}\cos\frac{2\pi p}{q} + \left\{S_n - \phi_{-n}\left(\frac{2}{q}-1\right)\right\}\cos\frac{4\pi p}{q}\right.$
$\left. + \left\{S_n - \phi_{-n}\left(\frac{3}{q}-1\right)\right\}\cos\frac{6\pi p}{q} + \cdots + \left\{S_n - \phi_{-n}\left(\frac{2-1}{q}-1\right)\right\}\cos\frac{(2q-2)\pi p}{q}\right]$

In the above two theorems p & q are integers.
In 16 & 17 the same theorems 14 & 15 are written in another form.

Ex. 1. $\phi_n\left(-\frac{1}{4}\right) - \phi_n\left(-\frac{3}{4}\right) = 2\cdot\dfrac{E_{n+1}}{4^{n+1}}\cos\frac{\pi n}{2}.$

Sol. Write $\frac{1}{2}$ for x in Th. 14.

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$$1 + \frac{(en)}{1^1} + \frac{(en)^2}{2^2} + \frac{(en)^3}{3^3} + \&c$$

$$= \sqrt{2\pi n}\, e^{n - \frac{1}{24n} - \frac{1}{48n^2} - \left(\frac{1}{36} + \frac{1}{5760}\right)\frac{1}{n^3} \cdots}$$

$$\int_0^\infty \frac{x^{n-1}\, dx}{1 + \left(\frac{x}{1}\right) + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{3}\right)^3 + \left(\frac{x}{4}\right)^4 + \&}$$

$$= n^n \left\{ \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \frac{3}{8n^4} + \&c \right\}$$

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2. $1^n - 3^n + 5^n - 7^n + \&c = \frac{1}{2} E_{n+1} \cos \frac{\pi n}{2}$.

3. $E_{1-n} \cos \frac{\pi n}{2} = \left(\frac{\pi}{2}\right)^n \frac{E_n}{\lfloor n-1}$.

4. $\pi \left\{ \frac{1}{2} - \frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} - \frac{1}{\sqrt{5}+\sqrt{7}} + \&c \right\}$

$= \frac{1}{1\sqrt{1}} - \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} - \frac{1}{7\sqrt{7}} + \&c.$

5. $\frac{\pi}{2} \left\{ \frac{1}{\sqrt{2}} + \frac{1}{2} - \frac{1}{\sqrt{1}+\sqrt{3}} - \frac{1}{\sqrt{2}+\sqrt{4}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{4}+\sqrt{6}} - \&c \right\}$

$= \frac{1}{1\sqrt{1}} + \frac{1}{5\sqrt{5}} + \frac{1}{9\sqrt{9}} + \frac{1}{13\sqrt{13}} + \&c$

6. If $\frac{p}{q}$ lies between 0 & 1 p being any integer & q an odd integer shew that

i. $\frac{(2\pi q)^n}{4\lfloor n-1} \left\{ \phi_{n-1}\left(\frac{p}{q} - 1\right) - \phi_{n-1}\left(\frac{p}{q}\right) \right\}$

$= \sin \frac{\pi n}{2} \left[\left\{ \phi_{-n}\left(\frac{1}{q} - 1\right) - \phi_{-n}\left(\frac{1}{q}\right) \right\} \sin \frac{2\pi p}{q} \right.$

$\left. + \left\{ \phi_{-n}\left(\frac{2}{q} - 1\right) - \phi_{-n}\left(-\frac{2}{q}\right) \right\} \sin \frac{4\pi p}{q} + \&c \text{ to } \frac{q-1}{2} \text{ terms} \right]$

ii. $\frac{(2\pi q)^n}{4\lfloor n-1} \left\{ \phi_{n-1}\left(\frac{p}{q} - 1\right) + \phi_{n-1}\left(-\frac{p}{q}\right) - 2S_{1-n}\left(1 - \frac{1}{q^n}\right) \right\}$

$= \cos \frac{\pi n}{2} \left[\left\{ \phi_{-n}\left(\frac{1}{q} - 1\right) - \phi_{-n}\left(-\frac{1}{q}\right) \right\} \cos \frac{2\pi p}{q} + \left\{ \phi_{-n}\left(\frac{2}{q} - 1\right) - \phi_{-n}\left(-\frac{1}{q}\right) \right\} \cos \frac{4\pi p}{q} \right.$

$\left. + \&c \text{ to } \frac{q-1}{2} \text{ terms} \right]$.

Hence shew that

7. $\frac{(6\pi)^n}{2\lfloor n-1 \sqrt{3}} \left\{ \phi_{n-1}\left(-\frac{1}{3}\right) - \phi_{n-1}\left(-\frac{1}{3}\right) \right\} = \left\{ \phi_{-n}\left(-\frac{1}{3}\right) - \phi_{-n}\left(-\frac{2}{3}\right) \right\} \sin \frac{\pi n}{2}$.

8. $\frac{2^{n-1}}{\lfloor n} \phi_n(-x) = \frac{\sin \pi x}{\pi^{n+1}} \cos\left(\pi x + \frac{\pi n}{2}\right) + \frac{\sin 2\pi x}{(2\pi)^{n+1}} \cos\left(2\pi x + \frac{\pi n}{2}\right) + \&c.$

Sol. Combine the results of IX 14 & 15.

9. $\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{1-x}} + \frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt{2-x}} + \frac{1}{\sqrt{2+x}} - \&c$

$= 2\left(\frac{\sin 2\pi x}{\sqrt{1}} + \frac{\sin 4\pi x}{\sqrt{2}} + \frac{\sin 6\pi x}{\sqrt{3}} + \&c \right)$

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$$(p-q-1)\int_0^\infty \frac{\left(1+\frac{x}{m}\right)^q}{\left(1+\frac{x}{n}\right)^p}\,dx$$

$$= \frac{1}{2}\cdot\frac{m^p}{n^q}\cdot\frac{\lfloor q}{\lfloor p-1}\cdot\frac{\lfloor p-q-1}{(m-n)^{p-q-1}}$$

$$+\frac{2}{3}(m+n)+mq-np$$

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CHAPTER X

1. $\frac{B_n}{n}\cos\frac{\pi n}{2} + \frac{1}{n}$ when n is 0 is a finite quantity which is invariably denoted by the Symbol C_0 the value of which can be found from X 2. It is the constant of S & its value is .5772156649015328606060... & $e^{-C_0} = .56145948356$.

Sol. Since L.H.S in IX 1 is finite when $n = -1$

$\frac{B_n}{n}\cos\frac{\pi n}{2} + \frac{x^n}{n}$ is finite when $n=0$ i.e

$\frac{B_n}{n}\cos\frac{\pi n}{2} + \frac{1}{2} + \frac{x^n-1}{n}$ is finite when $n=0$.

But $\frac{x^n-1}{n} = \log_e x$ a finite quantity when $n=0$

$\therefore \frac{B_n}{n}\cos\frac{\pi n}{2} + \frac{1}{n}$ is finite when $n=0$.

2. $1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{x} = \Sigma\frac{1}{x}$ or $\phi(x)$

$\Sigma\frac{1}{x} = C_0 + \log_e x + \frac{1}{2x} - \frac{B_2}{2x^2} + \frac{B_4}{4x^4} - \frac{B_6}{6x^6} + \frac{B_8}{8x^8} - \&c$

3. $\Sigma\frac{1}{x} = 1 - \frac{1}{x+1} + \frac{1}{2} - \frac{1}{x+2} + \frac{1}{3} - \frac{1}{x+3} + \&c$

$= \frac{x}{1(1+x)} + \frac{x}{2(2+x)} + \frac{x}{3(3+x)} + \&c$

4. $\Sigma\frac{(-1)^x}{x} = xS_2 - x^2S_3 + x^3S_4 - x^4S_5 + x^5S_6 - \&c$

5. $\Sigma\frac{1}{x-1} - \Sigma\frac{1}{-x} = -\pi\cot\pi x$.

Sol. Write πx for x in . Then we have

$\pi\cot\pi x = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \&c$

$= \left\{1 - \frac{1}{1-x} + \frac{1}{2} - \frac{1}{2-x} + \frac{1}{3} - \frac{1}{3-x} + \&\right\}$

$- \left\{1 - \frac{1}{x} + \frac{1}{2} - \frac{1}{1+x} + \frac{1}{3} - \frac{1}{2+x} + \&c\right\}$

$= \Sigma\frac{1}{-x} - \Sigma\frac{1}{x-1}$ by X 3.

6. $n\,\Sigma\frac{1}{x} - \left\{\Sigma\frac{1}{x/n} + \Sigma\frac{1}{\frac{x-1}{n}} + \Sigma\frac{1}{\frac{x-2}{n}} + \dots \Sigma\frac{1}{\frac{x-n+1}{n}}\right\}$

$= n\log n$.

Ex. 1. $\Sigma\frac{1}{x-\frac{1}{2}} = C_0 + \log_e x + \frac{2-1}{2}\cdot\frac{B_2}{2x^2} - \frac{2^3-1}{2^3}\cdot\frac{B_4}{4x^4} + \frac{2^5-1}{2^5}\cdot\frac{B_6}{6x^6} - \&c$

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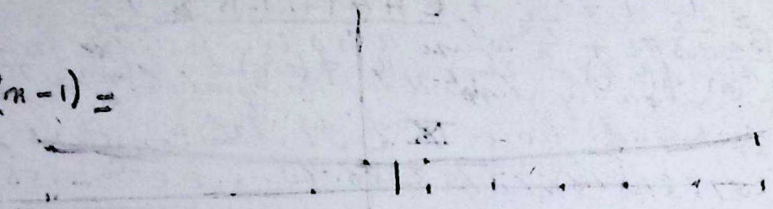
$$\frac{\lfloor \frac{n}{2} }{\pi^{\frac{n}{2}}} S_n (n-1) =$$



The maximum value of $\frac{a^x}{\lfloor x} = \frac{e^{\int \frac{x}{a} da}}{\sqrt{2\pi}}$

$$= \frac{a^{a-\frac{1}{2}}}{\lfloor a-\frac{1}{2}} e^{\frac{1}{32a(36a^2+10\cdot1)}}$$

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2. $\Sigma \frac{1}{-\frac{1}{n}} + \Sigma \frac{1}{-\frac{2}{n}} + \Sigma \frac{1}{-\frac{3}{n}} + \cdots \Sigma \frac{1}{-\frac{n-1}{n}} = -n \log_e n$

3. (a) $\phi(-\frac{1}{2}) = -2\log_e 2$; (b) $\phi(-\frac{2}{3}) = -\frac{3}{2}\log_e 3 - \frac{\pi}{2\sqrt{3}}$

(c) $\phi(-\frac{3}{4}) = -\frac{\pi}{2} - 3\log_e 2$ (d) $\phi(-\frac{5}{6}) = -\frac{\pi}{2}\sqrt{3} - 2\log_e 2 - \frac{3}{2}\log_e 3$

(e) $3\phi(-\frac{1}{2}) - 2\phi(-\frac{1}{3}) = \pi$.

4. $\phi(-\frac{1}{2n}) + \phi(-\frac{3}{2n}) + \cdots \phi(-\frac{2n-1}{2n}) = -n\log_e 4n$.

7. $\frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \cdots + \frac{1}{a+xb} = \frac{1}{b}\{\phi(\frac{a}{b}+x) - \phi(\frac{a}{b})\}$

8. $\frac{1}{a+b} - \frac{1}{a+2b} + \frac{1}{a+3b} - \&c = \frac{1}{2b}\{\phi(\frac{a}{2b}) - \phi(\frac{a-b}{2b})\}$

9. $\phi(\frac{1}{2x}) = \phi(\frac{1}{x}) - \log_e 2 + x\int_0^1 \frac{x^x}{1+x^x}dx$.

10. $\phi(-\frac{1}{x}) = -x\int_0^1 \frac{(1-x)^2}{x(x^x-1)}dx$.

11. $\phi(\frac{1}{x}-1) + \phi(-\frac{1}{x}) = -x\{1 + \frac{2}{x^3-x} + \frac{2}{(2x)^3-2x} + \&c\}$

12. $\frac{2}{x^3-x} + \frac{2}{(2x)^3-2x} + \frac{2}{(3x)^3-3x} + \&c = \int_0^1 \frac{x^{x-2}(1-x)^2}{1-x^x}dx$

13. $1 + \frac{2}{(2x)^3-2x} + \frac{2}{(4x)^3-4x} + \frac{2}{(6x)^3-6x} + \&c$

$= \frac{1}{2}\{1 + \frac{2}{x^3-x} + \frac{2}{(2x)^3-2x} + \&c\} + \log_e\frac{2}{x}$

+ Logmic part of $(1 - \frac{1}{1+x} + \frac{1}{1+2x} - \frac{1}{1+3x} + \&c)$.

14. (a) $\frac{x}{1} - \frac{x^{1+n}}{1+n} + \frac{x^{1+2n}}{1+2n} - \&c = \int_0^x \frac{1}{1+x^n}dx$

(b) $\frac{x}{1} + \frac{x^{1+n}}{1+n} + \frac{x^{1+2n}}{1+2n} + \&c = \int_0^x \frac{1}{1-x^n}dx$

(c) If n is odd $\int_0^x \frac{1}{1-x^n}dx = \int_0^x \frac{1}{1+(-x)^n}dx$

(d) If n is even $\int_0^x \frac{1}{1-x^n}dx = \frac{1}{2}\int_0^x \frac{1}{1+x^{\frac{n}{2}}}dx + \frac{1}{2}\int_0^x \frac{1}{1-x^{\frac{n}{2}}}dx$.

(e). If $l < n+1$

i. If n is even $\int \frac{x^{l-1}}{x^n-1}dx = \frac{1}{n}\log_e(x-1) + \frac{(-1)^l}{n}\log_e(x+1)$

$+ \frac{1}{n}\Sigma \cos\frac{rl\pi}{n}\log_e(x^2 - 2x\cos\frac{r\pi}{n} + 1)$

$- \frac{2}{n}\Sigma \sin\frac{rl\pi}{n}\tan^{-1}\frac{x - \cos\frac{r\pi}{n}}{\sin\frac{r\pi}{n}}$. $r = 2, 4, \ldots (n-2)$.

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$$\int_0^\infty \frac{x^{2m}}{(1+x^2)^{n+1}} \cos px\, dx = \frac{\pi}{2}(-1)^m \frac{e^{-p}}{2^n \lfloor n}$$

$$\times \left\{ p^n + A_1 p^{n-1} + A_2 p^{n-2} + A_3 p^{n-3} + \&c \right\}$$

$$A_n = \frac{\lfloor n+n}{\lfloor n-n} \cdot \frac{1}{2^n \lfloor n} \left\{ 1 - \frac{4}{1!} \cdot \frac{n\, m\, n}{(n+n)(n+n-1)} \right.$$

$$\left. + \frac{4^2}{2^2} \cdot \frac{n(n-1)\, m(m-1)\, n(n-1)}{(n+n)(n+n-1)(n+n-2)(n+n-3)} - \&c \right\}$$

$$\cot\theta + 4\left(\frac{\sin 2\theta}{e^{2y}-1} + \frac{\sin 4\theta}{e^{4y}-1} + \frac{\sin 6\theta}{e^{6y}-1} + \&c \right)$$

$$= \cancel{z \cot\phi \sqrt{1 - x\sin^2\phi}} + z \int_{\frac{2\theta z}{\pi}}^{\phi} \sqrt{1 - x\sin^2\phi}\, d\phi \cdot$$

$$- \frac{2\theta z}{\pi} \int_0^{\frac{\pi}{2}}$$

$$= z \left\{ \cot\phi \sqrt{1 - x\sin^2\phi} + \int_0^{\phi} \sqrt{1 - x\sin^2\phi}\, d\phi \right.$$

$$\left. - \frac{2\theta}{\pi} \int_0^{\frac{\pi}{2}} \sqrt{1 - x\sin^2\phi}\, d\phi \right\}$$

$$4\left(\frac{\sin 2\theta}{e^{4y} - e^{-y}} + \frac{\sin 4\theta}{e^{2y} - e^{-2y}} + \&c \right)$$

$$= z \left\{ \int_0^{\phi} \sqrt{1 - x\sin^2\phi}\, d\phi - \frac{2\theta}{\pi} \int_0^{\frac{\pi}{2}} \sqrt{1 - x\sin^2\phi}\, d\phi \right\}$$

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ii. If n be odd $\int \frac{x^{l-1}}{x^n-1}dx = \frac{1}{n}\log_e(x-1) + \frac{1}{n}\Sigma \cos\frac{r l\pi}{n}\log_e(x^2-2x\cos\frac{r\pi}{n}+1)$

$- \frac{2}{n}\Sigma \sin\frac{r l\pi}{n}\tan^{-1}\frac{x-\cos\frac{r\pi}{n}}{\sin\frac{\pi r}{n}}$. $r = 2, 4, 6 \ldots (n-1)$.

iii. If n be even $\int \frac{x^{l-1}}{x^n+1}dx = -\frac{1}{n}\Sigma \cos\frac{r l\pi}{n}\log_e(x^2-2x\cos\frac{r\pi}{n}+1)$

$+ \frac{2}{n}\sin\frac{\pi r l}{n}\tan^{-1}\frac{x-\cos\frac{\pi r}{n}}{\sin\frac{\pi r}{n}}$ $r = 1, 3, 5 \ldots (n-1)$.

iv. If n be odd $\int \frac{x^{l-1}}{x^n+1}dx = \frac{(-1)^{l-1}}{n}\log_e(x+1)$

$- \frac{1}{n}\Sigma \cos\frac{r l\pi}{n}\log_e(x^2-2x\cos\frac{r\pi}{n}+1)$

$+ \frac{2}{n}\Sigma \sin\frac{r l\pi}{n}\tan^{-1}\frac{x-\cos\frac{r\pi}{n}}{\sin\frac{\pi r}{n}}$ $r = 1, 3, 5, \ldots (n-1)$.

v. $\int_a^b f(x)dx = h\left\{\frac{1}{2}f(a) + \frac{1}{2}f(a) + f(a+h) + \ldots + f(b-h)\right\}$

$- \frac{h^2}{\lfloor 2}B_2\{f'(b) - f'(a)\} + \frac{h^4}{\lfloor 4}B_4\{f'''(b) - f'''(a)\} - \&c$

Exs. 1. $\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \&c = \log_e(1+x)$

2. $\frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \&c = \tan^{-1}x$

3. $\frac{x}{1} - \frac{x^4}{4} + \frac{x^7}{7} - \&c = \frac{1}{6}\log_e\frac{(1+x)^3}{1+x^3} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x\sqrt{3}}{2-x}$.

4. $\frac{x}{1} - \frac{x^5}{5} + \frac{x^9}{9} - \&c = \frac{1}{4\sqrt{2}}\log_e\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}$

5. $\frac{x}{1} - \frac{x^6}{6} + \frac{x^{11}}{11} - \&c = \frac{1}{20}\log_e\frac{(1+x)^5}{1+x^5}$.

$+ \frac{1}{4\sqrt{5}}\log_e\frac{1+x\cdot\frac{\sqrt{5}-1}{2}+x^2}{1-x\cdot\frac{\sqrt{5}-1}{2}+x^2} + \frac{1}{10}\sqrt{10-2\sqrt{5}}\tan^{-1}\frac{x\sqrt{10-2\sqrt{5}}}{4-x(\sqrt{5}+1)}$

$+ \frac{\sqrt{10+2\sqrt{5}}}{10}\tan^{-1}\frac{x\sqrt{10+2\sqrt{5}}}{4+x(\sqrt{5}-1)}$.

6. $\frac{x}{1} - \frac{x^7}{7} + \frac{x^{13}}{13} - \frac{x^{19}}{19} + \&c = \frac{1}{2}\tan^{-1}x + \frac{1}{6}\tan^{-1}x^3$

$+ \frac{1}{4\sqrt{3}}\log_e\frac{1+x\sqrt{3}+x^2}{1-x\sqrt{3}+x^2}$.

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If $\Pi(a) = (1+a)(1+ax)(1+ax^2)$ &c.

$$\frac{\Pi(a)\Pi(-b) - \Pi(-a)\Pi(b)}{\Pi(a)\Pi(-b) + \Pi(-a)\Pi(b)} = \frac{a-b}{1-x} + \frac{(a-bx)(ax-b)}{1-x^3} + \frac{x(a-bx^2)(ax^2-b)}{1-x^5} + \frac{x^2(a-bx^3)(ax^3-b)}{1-x^7} + \&c$$

$$\frac{(1-a^2x^4)(1-a^2x^8)(1-a^2x^{12})\,\&c}{(1-a^2x^2)(1-a^2x^6)(1-a^2x^{10})\,\&c} \cdot \frac{(1-b^2x^4)(1-b^2x^8)(1-b^2x^{12})\,\&c}{(1-b^2x^2)(1-b^2x^6)(1-b^2x^{10})\,\&c}$$

$$= \frac{1}{1-abx} + \frac{x(a-bx)(b-ax)}{(1+x^2)(1-abx)} + \frac{x(a-bx^3)(b-ax^3)}{(1+x^4)(1-abx)} +$$

$$\frac{(1-a^2x^6)(1-a^2x^7)(1-a^2x^{11})\,\&c}{(1-a^2x)(1-a^2x^5)(1-a^2x^9)\,\&c} \cdot \frac{(1-b^2x^3)(1-b^2x^7)(1-b^2x^{11})\,\&c}{(1-b^2x)(1-b^2x^5)(1-b^2x^9)\,\&c}$$

$$= \frac{1}{1-ab} + \frac{(a-bx)(b-ax)}{(1+x^2)(1-ab)} + \frac{(a-bx^3)(b-ax^3)}{(1+x^4)(1-ab)} + \&c$$

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7. $x - \frac{x^9}{9} + \frac{x^{17}}{17} - \frac{x^{25}}{25} + \&c =$

$$\frac{\sqrt{2+\sqrt{2}}}{16}\left\{\log_e \frac{1+x\sqrt{2+\sqrt{2}}+x^2}{1-x\sqrt{2+\sqrt{2}}+x^2} + i\tan^{-1}\frac{x\sqrt{2+\sqrt{2}}}{1-x^2}\right\}$$

$$+ \frac{\sqrt{2-\sqrt{2}}}{16}\left\{\log_e \frac{1+x\sqrt{2-\sqrt{2}}+x^2}{1-x\sqrt{2-\sqrt{2}}+x^2} + i\tan^{-1}\frac{x\sqrt{2-\sqrt{2}}}{1-x^2}\right\}$$

8. $x - \frac{x^{11}}{11} + \frac{x^{21}}{21} - \frac{x^{31}}{31} + \&c = \frac{1}{4}\tan^{-1}x - \frac{1}{20}\tan^{-1}x^5$

$$+ \frac{1}{4\sqrt{5}}\tan^{-1}\frac{(x-x^3)\sqrt{5}}{1-3x^2+x^4} + \frac{1}{40}\sqrt{10-2\sqrt{5}}\log_e \frac{1+\frac{x}{2}\sqrt{10-2\sqrt{5}}+x^2}{1-\frac{x}{2}\sqrt{10-2\sqrt{5}}+x^2}$$

$$+ \frac{1}{40}\sqrt{10+2\sqrt{5}}\log_e \frac{1+\frac{x}{2}\sqrt{10+2\sqrt{5}}+x^2}{1-\frac{x}{2}\sqrt{10-2\sqrt{5}}+x^2}.$$

(4). 1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \&c = \log_e 2$

2. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c = \frac{\pi}{4}$

3. $1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \&c = \frac{\pi}{3\sqrt{3}} + \frac{1}{3}\log_e 2$

4. $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \&c = \frac{\pi}{4\sqrt{2}} + \frac{1}{2\sqrt{2}}\log_e(1+\sqrt{2})$.

5. $1 - \frac{1}{8} + \frac{1}{11} - \frac{1}{16} + \&c = \frac{1}{5}\log_e 2 + \frac{1}{\sqrt{5}}\log_e \frac{1+\sqrt{5}}{2}$

$$+ \frac{\pi}{50}\left(2\sqrt{10-2\sqrt{5}} + \frac{1}{2}\sqrt{10+2\sqrt{5}}\right)$$

6. $1 - \frac{1}{7} + \frac{1}{13} - \frac{1}{19} + \&c = \frac{\pi}{6} + \frac{1}{\sqrt{3}}\log_e \frac{1+\sqrt{3}}{\sqrt{2}}$

7. $1 - \frac{1}{9} + \frac{1}{17} - \frac{1}{25} + \&c = \frac{\pi}{8\sqrt{2}}\sqrt{2+\sqrt{2}}$

$$+ \frac{1}{16}\sqrt{2+\sqrt{2}}\log_e \frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}} + \frac{1}{16}\sqrt{2-\sqrt{2}}\log_e \frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}.$$

8. $1 - \frac{1}{11} + \frac{1}{21} - \frac{1}{31} + \&c = \frac{\pi}{20} +$

$$\frac{\sqrt{10-2\sqrt{5}}}{40}\log_e \frac{4+\sqrt{10-2\sqrt{5}}}{4-\sqrt{10-2\sqrt{5}}} + \frac{\sqrt{10+2\sqrt{5}}}{40}\log_e \frac{4+\sqrt{10+2\sqrt{5}}}{4-\sqrt{10+2\sqrt{5}}}.$$

9. $1.\frac{1}{2} - \frac{1}{4}.\frac{1}{12} + \frac{1}{7}.\frac{1}{27} - \&c = \frac{\pi}{6\sqrt{3}} + \frac{1}{4}\log_e 3$

10. $\frac{\sqrt{3}-1}{1} - \frac{(\sqrt{3}-1)^4}{4} + \frac{(\sqrt{3}-1)^7}{7} - \&c = \frac{\pi}{4\sqrt{3}} + \frac{1}{3}\log_e \frac{1+\sqrt{3}}{\sqrt{2}}$.

11. $\frac{2-\sqrt{3}}{1} - \frac{(2-\sqrt{3})^3}{5} + \frac{(2-\sqrt{3})^5}{9} - \&c = \frac{\pi}{16}(\sqrt{3}-1) - \frac{\sqrt{3}-1}{4}\log_e(\sqrt{3}-1)$

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$$n - \frac{n}{\lfloor 1}(n+2)\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{u}{u+n+1}$$

$$+\frac{n(n+1)}{\lfloor 2}(n+4)\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$$\times\frac{u(u-1)}{(u+n+1)(u+n+2)} - \&c$$

$$= n\cdot\frac{\lfloor x+n \quad \lfloor y+n}{\lfloor n \quad \lfloor x+y+z}\left\{1+\frac{xy}{\lfloor 1}\cdot\frac{z+u+n+1}{(z+n+1)(u+n+1)}+\right.$$

$$\left.\frac{x(x-1)\,y(y-1)\,(z+u+n+1)(z+u+n+2)}{\lfloor 2 \quad (z+n+1)(z+n+2)(u+n+1)(u+n+2)}+\&c\right\}$$

$$\frac{\Pi(-a)\,\Pi(-d)}{\Pi(-b)\,\Pi(-c)}\left\{1+\frac{a-b}{1-x}\cdot\frac{a-c}{a-d}+\frac{(a-b)(a-bx)(a-c)(a-cx)}{(1-x)(1-x^2)(a-d)(a-dx)}+\&\right\}$$

$$=1+\frac{1-dx}{1-x}\cdot\frac{1-a}{a-d}\cdot\frac{b-d}{1-b}\cdot\frac{c-d}{1-c}+x^2\frac{(1-dx^3)(1-d)}{(1-x)(1-x^2)}\cdot\frac{(1-a)(1-ax)}{(a-d)(a-dx)}$$

$$\times\frac{(b-d)(b-dx)}{(1-b)(1-bx)}\cdot\frac{(c-d)(c-dx)}{(1-c)(1-cx)}+x^4\frac{(1-dx^5)(1-d)(1-dx)}{(1-x)(1-x^2)(1-x^3)}\cdot\frac{1-a}{a-d}$$

$$\times\frac{(1-ax)(1-ax^2)}{(a-dx)(a-dx^2)}\cdot\frac{(b-d)(b-dx)(b-dx^2)}{(1-b)(1-bx)(1-bx^2)}\cdot\frac{(c-d)(c-dx)(c-dx^2)}{(1-c)(1-cx)(1-cx^2)}+\&c$$

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C. 1. $1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\frac{2}{6^3-6}+\&c = 2\log_e 2$

2. $1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\frac{2}{9^3-9}+\&c = \log_e 3$

3. $1+\frac{2}{4^3-4}+\frac{2}{8^3-8}+\frac{2}{12^3-12}+\&c = \frac{3}{2}\log_e 2$

4. $1+\frac{2}{5^3-5}+\frac{2}{10^3-10}+\frac{2}{15^3-15}+\&c = \frac{1}{2}\log_e 5+\frac{1}{\sqrt{5}}\log_e\frac{\sqrt{5}+1}{2}$.

5. $1+\frac{2}{6^3-6}+\frac{2}{12^3-12}+\frac{2}{18^3-18}+\&c = \frac{1}{2}\log_e 3+\frac{1}{3}\log_e 4$.

6. $1+\frac{2}{8^3-8}+\frac{2}{16^3-16}+\frac{2}{24^3-24}+\&c = \log_e 2+\frac{\log_e(1+\sqrt{2})}{2\sqrt{2}}$.

7. $1+\frac{2}{10^3-10}+\frac{2}{20^3-20}+\frac{2}{30^3-30}+\&c = \frac{2}{5}\log_e 2+\frac{1}{4}\log_e 5$
$+\frac{3}{2\sqrt{5}}\log_e\left(\frac{1+\sqrt{5}}{2}\right)$.

8. $1+\frac{2}{12^3-12}+\frac{2}{24^3-24}+\frac{2}{36^3-36}+\&c = \frac{1}{2}\log_e 2+\frac{1}{4}\log_e 3$
$-\frac{1}{\sqrt{3}}\log_e(\sqrt{3}-1)$.

9. $1+\frac{2}{16^3-16}+\frac{2}{32^3-32}+\frac{2}{48^3-48}+\&c = \frac{5}{8}\log_e 2+\frac{1}{4\sqrt{2}}\log_e(1+\sqrt{2})$
$+\frac{\sqrt{2+\sqrt{2}}}{16}\log_e\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}+\frac{\sqrt{2-\sqrt{2}}}{16}\log_e\frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}$.

10. $1+\frac{2}{20^3-20}+\frac{2}{40^3-40}+\frac{2}{60^3-60}+\&c = \frac{1}{8}\log_e 5+\frac{3}{10}\log_e 2$
$+\frac{3}{4\sqrt{5}}\log_e\frac{\sqrt{5}+1}{2}+\frac{\sqrt{10-2\sqrt{5}}}{40}\log_e\frac{4+\sqrt{10-2\sqrt{5}}}{4-\sqrt{10-2\sqrt{5}}}$
$+\frac{\sqrt{10+2\sqrt{5}}}{40}\log_e\frac{4+\sqrt{10+2\sqrt{5}}}{4-\sqrt{10+2\sqrt{5}}}$.

15. If $\Sigma\frac{1}{x} = c_0+\log_e a$, then

$$\left(\frac{x+\frac{1}{2}}{a}\right)^{4x} = 1-\frac{n}{\underline{|1}}\cdot\frac{1}{6a^2}+\frac{n(n+1\frac{1}{10})}{\underline{|2}}\cdot\frac{1}{(6a^2)^2}-\frac{n(n^2+3\frac{3}{10}n+12\frac{51}{70})}{(6a^2)^3\underline{|3}}+\&c.$$

Cor. $\underline{|x}$ is minimum when $x=\frac{6}{13}$ very nearly.

Sol. $\underline{|x}$ is minimum when $\Sigma\frac{1}{x} = c_0+\log_e a = 1$

$\therefore x = \frac{1}{2}-\frac{1}{24}+\&c$ or $x = \frac{1}{2+\frac{1}{6}}$ very nearly.

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$$\left\{1 + a\cdot\frac{1-b}{1-x}\cdot\frac{1-c}{1-d} + a^2\cdot\frac{(1-b)(1-bx)}{(1-x)(1-x^2)}\cdot\frac{(1-c)(1-cx)}{(1-d)(1-dx)} + \&c\right\}$$

$$\times\left\{1 + \frac{a-b}{1-x}\cdot\frac{1-c}{1-d} + \frac{(a-b)(a-bx)}{(1-x)(1-x^2)}\cdot\frac{(1-c)(1-cx)}{(1-d)(1-dx)} + \&c\right\}$$

$$\times\frac{(1-a)(1-ax)(1-ax^2)\ \&c}{(1-b)(1-bx)(1-bx^2)\ \&c}$$

$$= 1 + \frac{a-b}{1-x}\cdot\frac{d-c}{1-d}\cdot\frac{1}{1-b}$$

$$+ \frac{(a-b)(a-bx)}{(1-x)(1-x^2)}\cdot\frac{(d-c)(dx-c)}{(1-d)(1-dx)}\cdot\frac{x}{(1-b)(1-bx)}$$

$$+ \frac{(a-b)(a-bx)(a-bx^2)}{(1-x)(1-x^2)(1-x^3)}\cdot\frac{(d-c)(dx-c)(dx^2-c)}{(1-d)(1-dx)(1-dx^2)}\cdot\frac{x^3}{(1-b)(1-bx)(1-bx^2)}$$

$$+ \&c$$

$$\frac{(1-ab)(1-abx)(1-abx^2)\ \&c}{(1-a)(1-ax)(1-ax^2)\ \&c}\cdot\frac{(1-ac)(1-acx)(1-acx^2)\ \&c}{(1-abc)(1-abcx)(1-abcx^2)\ \&c}$$

$$= 1 + a\cdot\frac{(1-b)(1-c)}{(1-a)(1-x)} + a^2\cdot\frac{(1-b)(x-b)\ (1-c)(x-c)}{(1-a)(1-ax)\ (1-x)(1-x^2)}$$

$$+ a^3\cdot\frac{(1-b)(x-b)(x^2-b)\ (1-c)(x-c)(x^2-c)}{(1-a)(1-ax)(1-ax^2)\ (1-x)(1-x^2)(1-x^3)} + \&c$$

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16. $C_0 = \log_e 2 - 1\left(\frac{2}{3^3-3}\right) - 2\left(\frac{2}{6^3-6} + \frac{2}{9^3-9} + \frac{2}{12^3-12}\right) - \&c$

the last term in the nth group being $\frac{2}{\left(\frac{3^n+3}{2}\right)^3 - \left(\frac{3^n+3}{2}\right)}$.

17. $\log_e \lfloor x = (x+\tfrac{1}{2})\log_e x - x + \tfrac{1}{2}\log_e(2\pi) + \frac{B_2}{1.2x} - \frac{B_4}{3.4x^3} + \&c$

Sol. Equate the coeff^{ts} of n in $\boxed{x}$ 1.

The coeff^t of n in $\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$ = that of n in

$-\frac{B_{n+1}}{n+1}\sin\frac{\pi n}{2}$ = that of n in $-\frac{\lfloor n}{\pi(2\pi)^n} S_{n+1}\sin\frac{\pi n}{2}$.

= that of n in $-\frac{n}{2}(1 - n\log_e 2\pi + \&c)\left(\frac{1}{n} + C_0 - \&c\right)(1 - nC_0 + \ldots)$

= that of n in $-\frac{1}{2} + \frac{n}{2}\log_e(2\pi) + \&c = \frac{1}{2}\log_e(2\pi)$

or as follows:—

Let c be the constant in $\log_e \lfloor x$ & let $f(x) = \log_e \frac{\lfloor x}{\lfloor \frac{x}{2} \lfloor \frac{x}{2}}$

then $f(x) - f(x-1) = \log_e 2$. $\therefore \log_e \frac{\lfloor x}{\lfloor \frac{x}{2} \lfloor \frac{x}{2}} = k + x\log_e 2$.

put $x=0$. then we see that $k = -\frac{1}{2}\log_e \pi$.

But the constant in $\log_e \frac{\lfloor x}{\lfloor \frac{x}{2} \lfloor \frac{x}{2}} = \frac{1}{2}\log_e 2 - c$.

$\therefore c = \frac{1}{2}\log_e(2\pi) = .9189\,385\,332\,046\,73$

Ex. Shew that when $x=\infty$ $\frac{e^x \lfloor x}{x^x \sqrt{2x+\frac{1}{3}}} = \sqrt{\pi}$.

18. $\log_e \lfloor x = -C_0 x + \frac{S_2}{2}x^2 - \frac{S_3}{3}x^3 + \frac{S_4}{4}x^4 - \&c$

i.e $\log_e \frac{\lfloor x+2}{2} = .922\,784\,3351 x + .197\,467\,0334 x^2$

$- .025\,685\,6344 x^3 + .004\,955\,8084 x^4$

$- .001\,135\,510 x^5 + .000\,286\,3437 x^6$

$- .000\,076\,6825 x^7 + .000\,021\,3883 x^8$

$- .000\,006\,1409 x^9 + .000\,005\,4047 \frac{x^{10}}{3+x}$.

Ex. 1. $\log_e \lfloor \frac{1}{3} = .5341\,990\,853$

$\log_e \lfloor \frac{1}{6} = .1211\,436\,313$

$\log_e \lfloor \frac{1}{10} = .0663\,762\,397$.

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$$\frac{1}{2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} + \frac{1}{1+\left(\frac{2}{x}\right)^2} + \cdots + \frac{1}{1+\left(\frac{x}{x}\right)^2}$$

$$= \frac{\pi x}{2x}\coth \pi x - \frac{\pi x}{4x} \cdot \frac{B_2}{2x1} \cdot \frac{1}{2} + \frac{B_6}{6x5} \cdot \frac{1}{2^3} - \frac{B_{10}}{10x9} \cdot \frac{1}{2^5} + \&c$$

$$\frac{(1-a)(1-ax)\,\&c\,(1-abc)(1-abcx)\,\&c\,(1-abd)(1-abdx)\,\&c\,(1-acd)(1-acdx)\,\&c}{(1-ab)(1-abx)\,\&c\,(1-ac)(1-acx)\,\&c\,(1-ad)(1-adx)\,\&c}$$

$$\times \frac{}{(1-abcd)(1-abcdx)\,\&c}$$

$$= 1 - a \cdot \frac{(1-b)(1-c)(1-d)}{(1-ab)(1-ac)(1-ad)} \cdot \frac{1-ax}{1-x} + a^2 \cdot \frac{(1-b)(x-b)(1-c)(x-c)}{(1-ab)(1-abx)(1-ac)(1-acx)}$$

$$\times \frac{(1-d)(x-d)}{(1-ad)(1-adx)} \cdot \frac{(1-ax^3)(1-a)}{(1-x)(1-x^2)} - a^3 \cdot \frac{(1-b)(x-b)(x^2-b)}{(1-ab)(1-abx)(1-abx^2)}$$

$$\times \frac{(1-c)(x-c)(x^2-c)(1-d)(x-d)(x^2-d)}{(1-ac)(1-acx)(1-acx^2)(1-ad)(1-adx)(1-adx^2)} \cdot \frac{(1-ax^5)(1-a)(1-ax)}{(1-x)(1-x^2)(1-x^3)}$$

$$+ \&c$$

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19. $\lfloor x-1 \rfloor -x = \pi \operatorname{cosec} \pi x$. Cor. $\lfloor -\tfrac{1}{2} = \sqrt{\pi}$.

20. $\lfloor \frac{x}{n} \lfloor \frac{x-1}{n} \lfloor \frac{x-2}{n} \cdots \lfloor \frac{x-n+1}{n} = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{x+\frac{1}{2}}} \lfloor x$.

Cor $\lfloor -\frac{1}{n} \lfloor -\frac{2}{n} \lfloor -\frac{3}{n} \lfloor -\frac{4}{n} \cdots \lfloor -\frac{n-1}{n} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{2\pi n}}$

Ex. 1. $\left(\lfloor -\tfrac{1}{2}\right)^2 = \lfloor -\tfrac{1}{3} \sqrt{\tfrac{\pi}{3}} \sqrt[3]{4}$.

2. $\lfloor i \lfloor -i = \frac{2\pi e^{\pi}}{e^{2\pi}-1}$.

3. $2^i \lfloor \tfrac{1}{2} i \lfloor \sqrt{\tfrac{1}{2}} i = \sqrt{\pi} \cdot i \lfloor i$.

4. $\frac{\lfloor x}{\lfloor \frac{x}{2} \lfloor \frac{x-1}{2}} = \frac{2^x}{\sqrt{\pi}}$.

5. $\log_e \lfloor x-\tfrac{1}{2} = x \log_e x - x + \log_e \sqrt{2\pi} + \frac{2-1}{2} \cdot \frac{B_2}{1.2x} - \frac{2^3-1}{2^3} \cdot \frac{B_4}{1.4x^3} + \&c.$

21. $\frac{\log_e 1}{1} + \frac{\log_e 2}{2} + \cdots + \frac{\log_e x}{x} = \phi(x)$

$\phi(x) = (\Sigma \tfrac{1}{x} - C_0) \log_e x - \tfrac{1}{2} (\log_e x)^2 + C_1 + \frac{B_2}{2x^2} \cdot 1 - \frac{B_4}{4x^4}(1+\tfrac{1}{2}+\tfrac{1}{3}) + \frac{B_6}{6x^6}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}) - \&c.$

$C_1 = -.0728158454883680$

Sol. Write $n-1$ for n in IX 1, then divide both sides by x^2 and find the coeff^{ts} of n from both sides & equate the

Ex. When $x = \infty$ shew that $\phi(x) - \tfrac{1}{2}(\Sigma \tfrac{1}{x} - C_0)^2 = C_1$.

22. $\phi(x) = \frac{\log_e 1}{1} - \frac{\log_e (1+x)}{1+x} + \frac{\log_e 2}{2} - \frac{\log_e (2+x)}{2+x} + \&c$

Cor. $\frac{\log_e 1}{1} - \frac{\log_e 3}{3} + \frac{\log_e 5}{5} - \frac{\log_e 7}{7} + \&c = \frac{\pi}{4} \log_e 2 + \tfrac{1}{4}\{\phi(-\tfrac{1}{4}) - \phi(-\tfrac{3}{4})\}$.

23. $n\phi(x) - \{\phi(\tfrac{x}{n}) + \phi(\tfrac{x-1}{n}) + \cdots + \phi(\tfrac{x-n+1}{n})\}$

$= n \log_e n (\Sigma \tfrac{1}{x} - C_0) - \frac{n}{2} (\log_e n)^2$

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$$\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots = \phi(x)$$

$$\phi\left(\frac{x}{1-y}\right) + \phi\left(\frac{y}{1-x}\right)$$
$$= \phi(x) + \phi(y) + \phi\left\{\frac{xy}{(1-x)(1-y)}\right\} + \log(1-x)\log(1-y).$$

$$\frac{1 + \frac{a-b}{1-x}\cdot\frac{1-c}{1-d} + \frac{(a-b)(a-bx)}{(1-x)(1-x^2)}\cdot\frac{(1-c)(1-cx)}{(1-d)(1-dx)} + \&c}{1 + \frac{c-d}{1-x}\cdot\frac{1-a}{1-b} + \frac{(c-d)(c-dx)}{(1-x)(1-x^2)}\cdot\frac{(1-a)(1-ax)}{(1-b)(1-bx)} + \&c}$$

$$= \frac{(1-b)(1-bx)(1-bx^2)\,\&c}{(1-a)(1-ax)(1-ax^2)\,\&c}\cdot\frac{(1-c)(1-cx)(1-cx^2)\,\&c}{(1-d)(1-dx)(1-dx^2)\,\&c}$$

$$\int_0^\infty \left\{\frac{x^x \lfloor n}{\lfloor n+x}} + e^{-x}\left(1+\frac{x}{n}\right)^n\right\} dx = \frac{e^n \lfloor n}{n^n} + \frac{6n}{12n+1}$$

very very nearly,

$$1 + \frac{n}{11} + \frac{n^2}{12^2} + \cdots + \frac{n^n}{12^n}\theta = \frac{e^n}{2};$$

where $\theta = \dfrac{4+15n}{8+45n}$ very nearly.

or still more approximately

$$\theta = \frac{2+7n\mu}{4+21n\mu}, \text{ where } \mu = \frac{466+585n}{480+576n}.$$

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Cor. $\phi(-\frac{1}{n}) + \phi(-\frac{2}{n}) + \phi(-\frac{3}{n}) + \cdots + \phi(-\frac{n-1}{n})$

$= n C_0 \log_e n + \frac{n}{2}(\log_e n)^2$

Ex. 1. $\frac{\sqrt{1}}{\sqrt{2}} \cdot \frac{\sqrt[3]{3}}{\sqrt[4]{4}} \cdot \frac{\sqrt[5]{5}}{\sqrt[6]{6}} \cdot \frac{\sqrt[7]{7}}{\sqrt[8]{8}} \cdot \frac{\sqrt[9]{9}}{\sqrt[10]{10}} \cdots$ ad.inf. $= 2^{\frac{1}{2}} \log_e 2 - C_0$.

2. $\phi(-\frac{1}{2}) = (\log_e 2)^2 + 2C_0 \log_e 2$

3. $\phi(-\frac{1}{3}) + \phi(-\frac{2}{3}) = \frac{3}{2}(\log_e 3)^2 + 3C_0 \log_e 3$.

4. $\phi(-\frac{1}{4}) + \phi(-\frac{3}{4}) = 7(\log_e 4)^2 + 6C_0 \log_e 2$.

5. $\phi(-\frac{1}{8}) + \phi(-\frac{5}{8}) = C_0(3\log_e 3 + 4\log_e 2) + \frac{3}{2}(\log_e 12)^2 - (\log_e 4)^2$.

24. $\frac{\pi}{2}\left\{\log_e \frac{\lfloor x-1}{\lfloor -x} + (C_0 + \log_e 2\pi)(2x-1)\right\}$ when x lies between 0 & 1.

$= \frac{\log_e 1}{1}\sin 2\pi x + \frac{\log_e 2}{2}\sin 4\pi x + \frac{\log_e 3}{3}\sin 6\pi x + \&c$.

N.B. $\frac{\pi}{2} - \pi x = \sin 2\pi x + \frac{1}{2}\sin 4\pi x + \frac{1}{3}\sin 6\pi x + \&c$

25. $\phi(x-1) - \phi(-x) = (C_0 + \log_e 2\pi)\pi \cot \pi x$ for the same limits

$+ 2\pi\left\{\sin 2\pi x \log_e 1 + \sin 4\pi x \log_e 2 + \&c\right\}$

N.B. $\sin 2\pi x + \sin 4\pi x + \sin 6\pi x + \&c = \frac{1}{2}\cot \pi x$.

Ex. 1. Find the values of $\phi(-\frac{1}{2})$, $\phi(-\frac{2}{3})$, $\phi(-\frac{3}{4})$ & $\phi(-\frac{5}{8})$.

2. $\frac{\log_e 1}{1} - \frac{\log_e 3}{3} + \frac{\log_e 5}{5} - \frac{\log_e 7}{7} + \&c = \frac{\pi}{4}\log_e \pi - \pi \log_e \lfloor -\frac{1}{4} - \frac{\pi}{4} C_0$.

3. $\frac{\left(\frac{\sqrt{1}}{\sqrt{2}} \cdot \frac{\sqrt[3]{3}}{\sqrt[4]{4}} \cdot \frac{\sqrt[5]{5}}{\sqrt[6]{6}} \cdot \frac{\sqrt[7]{7}}{\sqrt[8]{8}} \cdots\right)^{\frac{1}{\log_e 2}}}{\left(\frac{\sqrt{1}}{\sqrt[3]{3}} \cdot \frac{\sqrt[5]{5}}{\sqrt[7]{7}} \cdot \frac{\sqrt[9]{9}}{\sqrt[11]{11}} \cdot \frac{\sqrt[13]{13}}{\sqrt[15]{15}} \cdots\right)^{\frac{4}{\pi}}} = \frac{\sqrt{2}}{\pi}\left(\lfloor -\frac{1}{4}\right)^4$.

26. $(\log_e 1)^2 + (\log_e 2)^2 + (\log_e 3)^2 + \cdots + (\log_e x)^2 = x\psi(x)$

$\psi(x) = 2\log_e x \log_e \frac{\lfloor x}{\sqrt{2\pi}} - (x+\frac{1}{2})(\log_e x)^2 + 2x + \frac{1}{2}C_0^2 + C_1 - \frac{\pi^2}{24} - \frac{1}{2}(\log_e \pi)^2$

$+ 2\left\{\frac{B_4}{3.4} \cdot \frac{1+\frac{1}{2}}{x^3} - \frac{B_6}{5.6} \cdot \frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{x^5} + \&c\right\}$

Sol. Equate the coeff^{ts} of x^2 in IX.

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$$* \; 1 + \frac{\beta x}{(1-x)(1-\alpha x)} + \frac{\beta^2 x^4}{(1-x)(1-x^2)(1-\alpha x)(1-\alpha x^2)} \; - 4$$

$$\times (1-\alpha x)(1-\alpha x^2)(1-\alpha x^3) \; \&c$$

$$= 1 - \alpha x \frac{\alpha-\beta}{1-x} + x^3 \cdot \frac{(\alpha-\beta)(\alpha-\beta x)}{(1-x)(1-x^2)}$$

$$- x^6 \cdot \frac{(\alpha-\beta)(\alpha-\beta x)(\alpha-\beta x^2)}{(1-x)(1-x^2)(1-x^3)} + \&c \quad = F(\beta)$$

$$\frac{F(\beta)}{F(\beta x)} = 1 + \cfrac{\beta x}{1-\alpha x + \cfrac{\beta x^2}{1-\alpha x^2 + \cfrac{\beta x^3}{1-\alpha x^3 + \&c.}}}$$

where $* = F(\beta)$.

$$\cfrac{1}{1+\cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+}}}} \quad + \frac{a_{n+2}}{1} = \frac{N}{D}.$$

$$D = \phi_0(n) + \phi_1(n) + \phi_2(n) + \&c$$

& N is the D^o of $\cfrac{1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+}}} \quad \frac{a_{n+2}}{1}$

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27. $\psi(x) - \{\psi(\frac{x}{n}) + \psi(\frac{x-1}{n}) + \psi(\frac{x-2}{n}) + \dots + \psi(\frac{x-n+1}{n})\}$

$= 2\log_e n \log_e \frac{\lfloor x}{\sqrt{2\pi}} - x(\log_e n)^2 - (n-1)\{\frac{1}{2}C_0^2 + C_1 - \frac{\pi^2}{24} - \frac{1}{2}(\log_e 2\pi)^2\} - \frac{1}{2}(\log_e n)^2$

Cor. $\psi(-\frac{1}{n}) + \psi(-\frac{2}{n}) + \dots + \psi(-\frac{n-1}{n})$

$= \log_e n \log_e 2\pi + (n-1)\{\frac{1}{2}C_0^2 + C_1 - \frac{\pi^2}{24} - \frac{1}{2}(\log_e 2\pi)^2\} + \frac{1}{2}(\log_e n)^2$

Ex. 1. $\frac{1^{\frac{1}{1}} \cdot 2^{\frac{1}{2}} \cdot 3^{\frac{1}{3}} \cdot 4^{\frac{1}{4}} \dots x^{\frac{1}{x}}}{1^{\log 1} \cdot 2^{\log 2} \dots x^{\log x}} x^{x\log_e x - 2x} e^{2x + \frac{1}{2}(\Sigma\frac{1}{x} - \log_e x)^2}$

$= e^{\frac{\pi^2}{24}} (2\pi)^{\log_e \sqrt{2\pi}}$ when $x = \infty$.

2. $\psi(-\frac{1}{2}) = \frac{1}{2}C_0^2 + C_1 - \frac{\pi^2}{24} - \frac{1}{2}\{(\log_e 2)^2 + (\log_e \pi)^2\} + (\log_e 2)^2$

3. $\psi(-\frac{1}{3}) + \psi(-\frac{1}{3}) = C_0^2 + 2C_1 - \frac{\pi^2}{12} - (\log_e 2\pi)^2 + \log_e 3 \log_e 2\pi + \frac{1}{2}(\log_e 2)^2$

4. $\psi(-\frac{1}{4}) + \psi(-\frac{3}{4}) = C_0^2 + 2C_1 - \frac{\pi^2}{12} - \log_e \pi \log_e 2\pi + 1\frac{1}{2}(\log_e 2)^2$

5. $\frac{11}{2}\{\psi(-\frac{1}{6}) + \psi(-\frac{5}{6})\} = \frac{1}{2}C_0^2 + C_1 - \frac{\pi^2}{24} - \frac{1}{2}(\log_e 2\pi)^2 + \frac{1}{2}\log_e 2 \log_e 3.$

28. $\frac{\psi(x-1) + \psi(-x)}{2} = C_1 - \frac{\pi^2}{24} + \frac{1}{2}(C_0 + \log_e 2\pi)(C_0 - \log_e \frac{\pi}{2\sin^2 \pi x})$

$- \{\frac{\log_e 1}{1}\cos 2\pi x + \frac{\log_e 2}{2}\cos 4\pi x + \&c\}$

29. If C_n be the constant in $(\log 1)^n + (\log_e 2)^n + \dots (\log_e x)^n$ and if $\phi_n(x) = (\log 1)^n + (\log_e 2)^n + \dots \frac{1}{2}(\log_e x)^n - C_n$, then

i. The logarithmic part of $\phi_n(x) = n \log_e x\, \phi_{n-1}(x)$

$- \frac{n(n-1)}{\lfloor 2}(\log_e x)^2 \phi_{n-2}(x) + \frac{n(n-1)(n-2)}{\lfloor 3}(\log_e x)^3 \phi_{n-3}(x) - \&c.$

and the non-logarithmic part can be found from IX!

ii. $\phi_0(x)(\log_e x)^n - \frac{n}{1}\phi_1(x)(\log_e x)^{n-1} + \frac{n(n-1)}{\lfloor 2}\phi_2(x)(\log_e x)^{n-2} - \&c$

$= x\lfloor n - \frac{1}{x^n} \cdot \frac{B_{n+1}}{n+1}\sin\frac{\pi x}{2} - \frac{n}{2} \cdot \frac{1}{x^{n+1}} \cdot \frac{B_{n+2}}{n+2}\cos\frac{\pi x}{2}$

$+ \frac{n(n+1\frac{2}{3})}{2.4} \cdot \frac{1}{x^{n+2}} \cdot \frac{B_{n+3}}{n+3}\sin\frac{\pi x}{2} + \frac{n(n+2)(n+3)}{2.4.6} \cdot \frac{1}{x^{n+3}} \cdot \frac{B_{n+4}}{n+4}$

$\times \cos\frac{\pi x}{2}$

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$$1 + \frac{x^2}{(1-x)^2} + \frac{x^6}{(1-x)^2(1-x^2)^2} + \frac{x^{12}}{(1-x)^2(1-x^2)^2(1-x^3)^2} + \&c$$

$$= \frac{1 - x + x^3 - x^6 + x^{10} - \&c}{(1-x)(1-x^2)(1-x^3)(1-x^4)\,\&c}$$

$$\frac{(1+xy)(1+x^3y)(1+x^5y)\,\&c}{(1+\alpha xy)(1+\alpha x^3y)(1+\alpha x^5y)\,\&c} \cdot \frac{(1+\frac{x}{y})(1+\frac{x^3}{y})(1+\frac{x^5}{y})\,\&c}{(1+\beta\frac{x}{y})(1+\beta\frac{x^3}{y})(1+\beta\frac{x^5}{y})\,\&c}$$

$$\times \frac{(1-x^2)(1-x^4)(1-x^6)\,\&c}{(1-\alpha x^2)(1-\alpha x^4)(1-\alpha x^6)\,\&c} \cdot \frac{(1-\alpha\beta x^2)(1-\alpha\beta x^4)(1-\alpha\beta x^6)\,\&c}{(1-\beta x^2)(1-\beta x^4)(1-\beta x^6)\,\&c}$$

$$= 1 + \left\{xy \cdot \frac{1-\alpha}{1-\beta x^2} + \frac{x}{y} \cdot \frac{1-\beta}{1-\alpha x^2}\right\}$$

$$+ \left\{(xy)^2 \cdot \frac{(1-\alpha)(x^2-\alpha)}{(1-\beta x^2)(1-\beta x^4)} + \left(\frac{x}{y}\right)^2 \frac{(1-\beta)(x^2-\beta)}{(1-\alpha x^2)(1-\alpha x^4)}\right\}$$

$$+ \left\{(xy)^3 \cdot \frac{(1-\alpha)(x^2-\alpha)(x^4-\alpha)}{(1-\beta x^2)(1-\beta x^4)(1-\beta x^6)} + \left(\frac{x}{y}\right)^3 \frac{(1-\beta)(x^2-\beta)(x^4-\beta)}{(1-\alpha x^2)(1-\alpha x^4)(1-\alpha x^6)}\right\}$$

$$+ \&c \quad \&c \quad \&c$$

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$$- \frac{n(n+2)(n+4)^2 + \frac{n(n+2)}{3} + \frac{4n}{5}}{2.4.6.8} \cdot \frac{1}{x^{n+4}} \cdot \frac{\beta_{n+5}}{n+5} \sin \frac{\pi n}{2}$$

$$- \frac{n(n+4)(n+5)\left\{(n+3)(n+4) + \frac{2}{3}(n+1)\right\}}{2.4.6.8.10} \cdot \frac{1}{x^{n+5}} \cdot \frac{\beta_{n+6}}{n+6} \cos \frac{\pi n}{2} + \&c.$$

iii. $\phi_n\left(\frac{x}{n}\right) + \phi_n\left(\frac{x-1}{n}\right) + \cdots + \phi_n\left(\frac{x-n+1}{n}\right)$

$= \phi_n(x) - n \log_e n \, \phi_{n-1}(x) + \frac{n(n-1)}{\lfloor 2} (\log n)^2 \phi_{n-2}(x) - \&c$

Cor 1. $\phi_n\left(-\frac{1}{n}\right) + \phi_n\left(-\frac{2}{n}\right) + \cdots + \phi_n\left(-\frac{n-1}{n}\right)$

$= -\left\{C_n - n \log_e n \, C_{n-1} + \frac{n(n-1)}{\lfloor 2} (\log n)^2 C_{n-2} - \&c\right\}$

Cor 2. There will be no logarithmic function in

$\phi_n\left(\frac{1}{x}\right) + \phi_n\left(\frac{2}{x}\right) + \cdots + \phi_n\left(\frac{x}{x}\right)$.

30. Let $1^n + 2^n/k + 3^n/k^2 + \cdots + x^n/k^{x-1} = k^x \phi(x) = F_k(x)$, then

i. $\phi(x) = C_n(k) + x^n \frac{\psi_0(-k)}{k-1} - \frac{n}{\lfloor 1} x^{n-1} \frac{\psi_1(-k)}{(k-1)^2} +$

$\frac{n(n-1)}{\lfloor 2} \cdot \frac{\psi_2(-k)}{(k-1)^3} - \&c$ where ψ is the same ψ in

ii. $C_n(k) = \frac{\psi_n(-k)}{(1-k)^{n+1}}$ &c $k \psi_n(-k) = k^n \psi\left(-\frac{1}{k}\right)$.

iii. $F_k\left(\frac{x}{n}\right) + F_k\left(\frac{x-1}{n}\right) + \cdots + F_k\left(\frac{x-n+1}{n}\right) - n C_n(k)$

$= \frac{\sqrt[n]{k}}{k n^n} \left\{F_{\sqrt[n]{k}}(x) - C_n(\sqrt[n]{k})\right\}$

Cor. $F_k\left(-\frac{1}{n}\right) + F_k\left(-\frac{2}{n}\right) + \cdots + F_k\left(-\frac{n-1}{n}\right) = n C_n(k) - \frac{\sqrt[n]{k} \, C_n(\sqrt[n]{k})}{k n^n}$.

31. Let $\frac{\log 1}{1^n} + \frac{\log 2}{2^n} + \frac{\log 3}{3^n} + \cdots + \frac{\log x}{x^n} = \phi_n(x)$ & let C_n' be the constant. Then,

i. $\phi_n(x) = C_n' - \left\{\frac{1}{(x+1)^n} + \frac{1}{(x+2)^n} + \frac{1}{(x+3)^n} + \&c\right\} \log_e x - \frac{1}{(n-1)^2} \cdot \frac{1}{x^{n-1}}$

$+ \beta_2 \frac{n}{\lfloor 2} \cdot \frac{1}{n x^{n+1}} - \beta_4 \frac{n(n+1)(n+2)}{\lfloor 4} \left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}\right) \frac{1}{x^{n+3}}$

$+ \beta_6 \frac{n(n+1)(n+2)(n+3)(n+4)}{\lfloor 6} \left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4}\right) \frac{1}{x^{n+5}}$ &c

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If $\int_{\alpha_1}^{\beta_1} \phi_1(p,x)\, F(nx)\, dx = \psi_1(p,n)$

& $\int_{\alpha_2}^{\beta_2} \phi_2(p,x)\, F(nx)\, dx = \psi_2(p,n)$

then $\int_{\alpha_1}^{\beta_1} \phi_1(p,x)\, \psi_2(q,nx)\, dx = \int_{\alpha_2}^{\beta_2} \phi_2(q,x)\, \psi_1(p,x)$

$$\int_{-\infty}^{\infty} \frac{e^{mx} \begin{matrix}\cos\\ \sin\end{matrix}(px)}{(1+e^x)^n}\, dx$$

$$= \frac{\lfloor m-1 \rfloor \lfloor n-m-1}{\lfloor n-1}\cdot \begin{matrix}\cos\\ \sin\end{matrix}\left(\tan^{-1}\frac{p}{n-m} - \tan^{-1}\frac{p}{m} + \tan^{-1}\frac{p}{n-m+1}\right.$$

$$\left. - \tan^{-1}\frac{p}{m+1} + \&c\right)$$

$$\div \sqrt{\left\{1+\frac{p^2}{(n-m)^2}\right\}\left\{1+\frac{p^2}{m^2}\right\}\left\{1+\frac{p^2}{(n-m+1)^2}\right\}\left\{1+\frac{p^2}{(m+1)^2}\right\}\&c}$$

$$\int_0^{\infty} e^{-mx}(1-e^{-x})^n \begin{matrix}\cos\\ \sin\end{matrix}(px)\, dx$$

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ii. $\phi_n(x) = n x c'_{n+1} - \frac{n(n+1)}{\lfloor 2} x^2 c'_{n+2} + \frac{n(n+1)(n+2)}{\lfloor 3} x^3 c'_{n+3} - \&c$

$- n \cdot \frac{1}{n} x S_{n+1} + \frac{n(n+1)}{\lfloor 2} \left(\frac{1}{n} + \frac{1}{n+1}\right) x^2 S_{n+2} - \&c.$

iii. $n^n \phi_n(x) - \left\{ \phi_n\left(\frac{x}{n}\right) + \phi_n\left(\frac{x-1}{n}\right) + \cdots + \phi_n\left(\frac{x-n+1}{n}\right) \right\}$

$= c'_n (n^n - n) - n^n \log_e n \left\{ \frac{1}{(x+1)^n} + \frac{1}{(x+2)^n} + \frac{1}{(x+3)^n} + \&c \right\}$

Cor. $\phi_n\left(-\frac{1}{n}\right) + \phi_n\left(-\frac{2}{n}\right) + \phi_n\left(-\frac{3}{n}\right) + \cdots + \phi_n\left(-\frac{n-1}{n}\right)$

$= n^n \log_e n \, S_n - (n^n - n) c'_n.$

32. Let $(\log_e 1)^n + \frac{1}{2}(\log_e 2)^n + \frac{1}{3}(\log_e 3)^n + \&c$ to x terms $= \phi_n(x)$ & let C_n be its constant. then

i. $\phi_n(x) - \frac{1}{n+1}(\log_e x)^{n+1} = C_n$ when $x = \infty$

ii. $n \phi_n(x) - \left\{ \phi_n\left(\frac{x}{n}\right) + \phi_n\left(\frac{x-1}{n}\right) + \phi_n\left(\frac{x-2}{n}\right) + \cdots + \phi_n\left(\frac{x-n+1}{n}\right) \right\}$.

$= \frac{n}{n+1} (\log_e n)^{n+1} \cos \pi n + n n \log_e n \left\{ \phi_{n-1}(x) - C_{n-1} \right\} -$

$\frac{n(n-1)}{\lfloor 2} n (\log_e n)^2 \left\{ \phi_{n-2}(x) - C_{n-2} \right\} + \&c$ the last term being

$(-1)^{n-1} n (\log_e n)^2 \left\{ \phi_0(x) - C_0 \right\}$

33. $\frac{(\log_e 1)^n}{1^{n+1}} + \frac{(\log_e 2)^2}{2^{n+1}} + \frac{(\log_e 3)^2}{3^{n+1}} + \&c$

$= \frac{\lfloor n}{n^{n+1}} + C_n - \frac{n}{\lfloor 1} C_{n+1} + \frac{n^2}{\lfloor 2} C_{n+2} - \frac{n^3}{\lfloor 3} C_{n+3} + \&c$

Sol. Differentiate n times both sides in

Ex. Shew that

1. $\frac{(\log_e 1)^3}{1\sqrt{1}} + \frac{(\log_e 2)^3}{2\sqrt{2}} + \frac{(\log_e 3)^3}{3\sqrt{3}} + \&c = 96.001$ nearly.

2. $\frac{\log_e 1}{1^2} + \frac{\log_e 2}{2^2} + \frac{\log_e 3}{3^2} + \&c = .9382$ nearly.

3. $\frac{(\log_e 1)^4}{1^2} + \frac{(\log_e 2)^4}{2^2} + \frac{(\log_e 3)^4}{3^2} + \&c = 24$ nearly.

4. $\frac{(\log_e 1)^5}{1\sqrt{1}} + \frac{(\log_e 2)^5}{2\sqrt{2}} + \frac{(\log_e 3)^5}{3\sqrt{3}} + \&c = 7680$ nearly.

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If $\int_0^\infty \phi_1(x) F(nx)\,dx = \psi_1(n)$

& $\int_0^\infty \phi_2(x) F(nx)\,dx = \psi_2(n)$

then $\int_0^\infty \phi_1(ax)\,\psi_2(bx)\,dx = \int_0^\infty \phi_2(ax)\,\psi_1(bx)$

If $\int_{\alpha_1}^{\beta_1} \phi_1(x) F(nx)\,dx = \psi_1(n)$

& $\int_{\alpha_2}^{\beta_2} \phi_2(x) F(ax)\,dx = \psi_2(n)$ } then

$\int_{\alpha_1}^{\beta_1} \phi_1(x)\,\psi_2(ax)\,dx = \int_{\alpha_2}^{\beta_2} \phi_2(x)\,\psi_1(ax)\,dx$

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5. $\frac{(\log_e 1)^5}{1^2}\sqrt{\log_e 1} + \frac{(\log_e 2)^5}{2^2}\sqrt{\log_e 2} + \frac{(\log_e 3)^5}{3^2}\sqrt{\log_e 3} + \&c = 288$ nearly.

34. $\frac{\log_e 1}{\sqrt{1}} + \frac{\log_e 2}{\sqrt{2}} + \frac{\log_e 3}{\sqrt{3}} + \cdots + \frac{\log_e x}{\sqrt{x}} = \phi(x)$

i $\phi(x) = \frac{\log_e 1}{\sqrt{1}} - \frac{\log_e(1+x)}{\sqrt{1+x}} + \frac{\log_e 2}{\sqrt{2}} - \frac{\log_e(2+x)}{\sqrt{2+x}} + \&c$

ii. $\phi(x) = \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{x}}\right)\log_e x$

$+ (\sqrt{2}+1)\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)\left(\log_e x + \frac{1}{2}C_0 + \frac{\pi}{4} + \log_e\sqrt{8\pi}\right)$

$- 4\sqrt{x} + \frac{1}{2}\cdot\frac{B_2}{1\sqrt{x}} - \frac{1.3.5}{1.4.6}\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{B_4}{2x^3\sqrt{x}}$

$+ \frac{1.3.5.7.9}{2.4.6.8.10}\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}\right)\frac{B_6}{3x^5\sqrt{x}} - \&c.$

iii. $\phi(x) - \frac{1}{\sqrt{n}}\left\{\phi\left(\frac{x}{n}\right) + \phi\left(\frac{x-1}{n}\right) + \cdots + \phi\left(\frac{x-n+1}{n}\right)\right\}$

$= \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{x}}\right)\log_e n$

$- (1+\sqrt{2})\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)\left\{(\sqrt{n}-1)\left(\frac{1}{2}C_0 + \frac{\pi}{4} + \log_e\sqrt{8\pi}\right) - \log_e n\right\}.$

iv. If $\psi(x) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{x}}$, then

$\left\{\phi(x-1) + \phi(-x) - 2c\right\} + \left(C_0 + \frac{\pi}{2} + \log_e 8\pi\right)\left\{\psi(x-1) + \psi(-x) - 2c'\right\}$

$= 2\left\{\frac{\log_e 1}{\sqrt{1}}\cos 2\pi x + \frac{\log_e 2}{\sqrt{2}}\cos 4\pi x + \&c\right\}$

v. $\left\{\phi(x-1) - \phi(-x)\right\} + \left(C_0 - \frac{\pi}{2} + \log_e 8\pi\right)\left\{\psi(x-1) - \psi(-x)\right\}$

$= 2\left\{\frac{\log_e 1}{\sqrt{1}}\sin 2\pi x + \frac{\log_e 2}{\sqrt{2}}\sin 4\pi x + \&c\right\}.$

In both cases c & c' are the constants of $\phi(x)$ & $\psi(x)$ respectively

Ex. 1. Find the values of $\phi(\frac{1}{2})$, $\phi(-\frac{2}{3})$ & $\phi(-\frac{3}{4})$.

2. Shew that the constant in $\phi(x) = -\frac{1}{2}S_{\frac{1}{2}}\left(C_0 + \frac{\pi}{2} + \log_e 8\pi\right)$.

$= 3.92265\ldots = 2\left\{2 - \frac{1}{2}\cdot\frac{B_{\frac{1}{2}}}{2} + \frac{1.3.5}{1.4.6}\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{B_{\frac{1}{4}}}{4} - \&c\right\}$

Sol. Write $\frac{1+h}{2}$ for n in IX L. and equate the coeff^{ts} of h.

Put $x=1$ in XI 34 ii. Then the 2nd result is at once obtained

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$$\phi(1)+\phi(2)+\cdots+\phi(h)$$
$$= \int_0^h \phi(x)\,dx + \tfrac{1}{2}\phi(h) + \int_0^\infty \frac{\phi(h+xi)-\phi(h-xi)}{i(e^{2\pi x}-1)}\,dx$$

The imaginary part in

$$\phi(n)F(a+bi)-\phi(2n)F(a+2bi)+\phi(3n)F(a+3bi)-\&c$$
$$= \int_0^\infty \frac{F(a+bx)-F(a-bx)}{e^{\pi x}-e^{-\pi x}}\,\phi(b^2x^2)\,dx.$$

$$\int_0^\infty \frac{e^{ax}-e^{-ax}}{e^{\pi x}-e^{-\pi x}}\cdot\frac{dx}{1+x^2x^2}$$

$$= \frac{\sin a}{1+n} - \frac{\sin 2a}{1+2n} + \frac{\sin 3a}{1+3n} - \&c.$$

a lying between 0 & π.

$$\operatorname{cosec}\phi + 2\left\{\frac{\sin\theta}{e^{2\pi}+1} + \frac{\sin 3\theta}{e^{4\pi}+1} + \frac{\sin 5\theta}{e^{6\pi}+1}\right\} =$$

$$= 2\operatorname{cosec}\phi$$

$$\frac{\cos\theta}{\sinh\frac{\pi}{2}} - \frac{\cos 3\theta}{\sinh\frac{3\pi}{2}} + \frac{\cos 5\theta}{\sinh\frac{5\pi}{2}} - \&c$$

$$= \frac{\pi}{2\sqrt{2}}\cdot\frac{\cos\phi}{\sqrt{1-\tfrac{1}{2}\sin^2\phi}}$$

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CHAPTER XI.

1. If $S_n = \frac{1}{(1-a)^n} - \frac{1}{(1+a)^n} + \frac{1}{(3-a)^n} - \frac{1}{(3+a)^n} + \&c$, then

i. If n is odd,

$$\frac{\cos(1-a)x}{(1-a)^n} - \frac{\cos(1+a)x}{(1+a)^n} + \frac{\cos(3-a)x}{(3-a)^n} - \frac{\cos(3+a)x}{(3+a)^n} + \&c$$

$= S_n - \frac{x^2}{\lfloor 2} S_{n-2} + \frac{x^4}{\lfloor 4} S_{n-4} - \&c$ as far as the term containing S_1.

ii. If n is even,

$$\frac{\sin(1-a)x}{(1-a)^n} - \frac{\sin(1+a)x}{(1+a)^n} + \frac{\sin(3-a)x}{(3-a)^n} - \frac{\sin(3+a)x}{(3+a)^n} + \&c$$

$= \frac{x}{\lfloor 1} S_{n-1} - \frac{x^3}{\lfloor 3} S_{n-3} + \frac{x^5}{\lfloor 5} S_{n-5} - \&c$ as far as the term containing S_1.

2. If $S_n = \frac{1}{(1-a)^n} + \frac{1}{(1+a)^n} + \frac{1}{(3-a)^n} + \frac{1}{(3+a)^n} + \&c$, then

i. If n is even

$$\frac{\cos(1-a)x}{(1-a)^n} + \frac{\cos(1+a)x}{(1+a)^n} + \frac{\cos(3-a)x}{(3-a)^n} + \frac{\cos(3+a)x}{(3+a)^n} + \&c$$

$= S_n - \frac{x^2}{\lfloor 2} S_{n-2} + \frac{x^4}{\lfloor 4} S_{n-4} - \&c$ as far as the term containing S_2.

ii. If n is odd

$$\frac{\sin(1-a)x}{(1-a)^n} + \frac{\sin(1+a)x}{(1+a)^n} + \frac{\sin(3-a)x}{(3-a)^n} + \frac{\sin(3+a)x}{(3+a)^n} + \&c$$

$= \frac{x}{\lfloor 1} S_{n-1} - \frac{x^3}{\lfloor 3} S_{n-3} + \frac{x^5}{\lfloor 5} S_{n-5} - \&c$ as far as the term containing S_1.

Sol. In both 1 & 2 expand the series in ascending powers of x and apply

3. If $\phi(n) = \frac{\cos x}{1^n} - \left(1 + \frac{1}{2}\right)\frac{\cos 3x}{3^n} + \left(1 + \frac{1}{2} + \frac{1}{3}\right)\frac{\cos 5x}{5^n} - \&c$, then

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$$\int_0^\infty e^{-x} \frac{x^p}{\lfloor p} \left\{ \int_0^\infty \frac{e^{-y}}{x+y} \cdot \frac{y^{p+q-1}}{\lfloor p+q-1} \, dy \right\}^n dx$$

$$= \frac{1}{q^n + p\left(\frac{\lfloor q-1}{\lfloor q-n} - q^{n-1}\right)}$$

$$\frac{1}{\sin^2\theta} - 8\left(\frac{1\cos 2\theta}{e^{2\pi}-1} + \frac{2\cos 4\theta}{e^{4\pi}-1} + \frac{3\cos 6\theta}{e^{6\pi}-1} + \dots\right)$$

$$= \frac{z^2}{\sin^2\phi} - z^2\left(\frac{1+x}{3}\right) + \frac{1}{3}\left\{1 - 24\left(\frac{1}{e^{2\pi}-1} + \frac{2}{e^{4\pi}-1} + \dots\right)\right\}$$

$$\frac{\cos\theta}{\sin^3\theta} - 8\left(\frac{1^2\sin 2\theta}{e^{2\pi}-1} + \frac{2^2\sin 4\theta}{e^{4\pi}-1} + \dots\right)$$

$$= z^3 \frac{\cos\phi}{\sin^3\phi}\sqrt{1 - x\sin^2\phi}$$

where $\theta z = \int_0^\phi \frac{d\phi}{\sqrt{1 - x\sin^2\phi}}$.

$$\sec\theta + 4\left\{\frac{\cos\theta}{e^{2\pi}-1} - \frac{\cos 3\theta}{e^{6\pi}-1} + \frac{\cos 5\theta}{e^{10\pi}-1} - \dots\right\}$$

$$= z\sec\phi\sqrt{1 - x\sin^2\phi}$$

$$\frac{\sin\theta}{\cosh\frac{1}{2}} - \frac{\sin 3\theta}{\cosh\frac{3}{2}} + \frac{\sin 5\theta}{\cosh\frac{5}{2}} - \dots =$$

$$= \frac{z}{2}\sqrt{x(1-x)}\cdot\frac{\sin\phi}{\sqrt{1 - x\sin^2\phi}}$$

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If n is odd $\phi(n-2) - \phi(n) =$

$$x\left\{\left(\frac{\sin x}{1^{n-2}} - \frac{\sin 3x}{3^{n-2}} + \frac{\sin 5x}{5^{n-2}} - \&c\right) - \left(\frac{\sin x}{1^n} - \frac{\sin 3x}{3^n} + \frac{\sin 5x}{5^n} - \&c\right)\right\}$$

$$+n\left\{\left(\frac{\cos x}{1^{n-1}} - \frac{\cos 3x}{3^{n-1}} + \frac{\cos 5x}{5^{n-1}} - \&c\right) - \left(\frac{\cos x}{1^{n+1}} - \frac{\cos 3x}{3^{n+1}} + \frac{\cos 5x}{5^{n+1}} - \&c\right)\right\}.$$

4. i. $\frac{1}{2}\sin 2x - \frac{1+\frac{1}{2}}{3}\sin 3x + \frac{1+\frac{1}{2}+\frac{1}{3}}{4}\sin 4x - \&c$

$= \frac{x}{2}\left(\frac{\cos x}{1} - \frac{\cos 2x}{2} + \frac{\cos 3x}{3} - \&c\right)$

ii. $\cos 2x - \left(1+\frac{1}{3}\right)\cos 4x + \left(1+\frac{1}{3}+\frac{1}{5}\right)\cos 6x - \&c$

$= \frac{\pi}{4}(\cos x - \cos 3x + \cos 5x - \&c)$

Integrate both sides in the above results n times.

5. i. $\sin a\theta + \frac{n}{\lfloor 1}\sin(a+1)\theta + \frac{n(n-1)}{\lfloor 2}\sin(a+2)\theta + \&c$

$= (2\cos\theta)^n \sin(a+n)\theta.$

ii. $\cos a\theta + \frac{n}{\lfloor 1}\cos(a+1)\theta + \frac{n(n-1)}{\lfloor 2}\cos(a+2)\theta + \&c$

$= (2\cos\theta)^n \cos(a+n)\theta$

Sol. find $e^{a+bi} + \frac{n}{\lfloor 1} e^{(a+1)bi} + \frac{n(n-1)}{\lfloor 2} e^{(a+2)bi} + \&c$

and apply the De. moivre's theorem.

6. i. $\sin x - \frac{1}{2}\sin 3x + \frac{1\cdot 3}{2\cdot 4}\sin 5x - \&c =$ I

$\frac{1}{2}\sin 2x - \frac{1\cdot 3}{2\cdot 4}\sin 4x + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin 6x - \&c = \frac{\sin\frac{x}{2}}{\sqrt{2}\cos x}$ II

ii. $\cos x - \frac{1}{2}\cos 3x + \frac{1\cdot 3}{2\cdot 4}\cos 5x - \&c$ I

$= 1 - \frac{1}{2}\cos 2x + \frac{1\cdot 3}{2\cdot 4}\cos 4x - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos 6x + \&c = \frac{\cos\frac{x}{2}}{\sqrt{2}\cos x}$ II

iii. $\sin 2x - \frac{1}{2}\sin 4x + \frac{1\cdot 3}{2\cdot 4}\sin 6x - \&c = \frac{\sin\frac{3x}{2}}{\sqrt{2}\cos x}$

iv. $\cos 2x - \frac{1}{2}\cos 4x + \frac{1\cdot 3}{2\cdot 4}\cos 6x - \&c = \frac{\cos\frac{3x}{2}}{\sqrt{2}\cos x}$

v. $\frac{\sin 2x}{2} - \frac{1}{2}\cdot\frac{\sin 4x}{4} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\sin 6x}{6} - \&c = \sin\frac{\pi}{2}\sqrt{2}\cos x$

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$$\frac{n}{n+}\,\frac{x}{n+1+}\,\frac{x}{n+2+}\cdots\cdots+\frac{x}{n+n}.$$

$$= 1 + \frac{n\,x}{\lfloor 1\, n(n+n)} + \frac{(n-1)(n-2)\,x^2}{\lfloor 2\, n(n+1)(n+n)(n+n-1)}$$

$$+\frac{(n-2)(n-3)(n-4)\,x^3}{\lfloor 3\, n(n+1)(n+2)(n+n)(n+n-1)(n+n-2)} + \&c \Big\}$$

$$= \Big\{1 + \frac{(n-1)\,x}{\lfloor 1\,(n+1)(n+n)} + \frac{(n-2)(n-3)\,x^2}{\lfloor 2\,(n+1)(n+2)(n+n)(n+n-1)} + \&c\Big\}$$

no. of terms being limited.

$$\frac{1}{1+}\,\frac{x}{1+}\,\frac{x}{1+}\,\frac{2x}{1+}\,\frac{2x}{1+}\cdots\cdots+\frac{(n-1)\,x}{1+}\,\frac{nx}{1}.$$

$$= D = 1 + \frac{n^2}{\lfloor 1} x + \frac{n^2(n-1)^2}{\lfloor 2} x^2 + \frac{n^2(n-1)^2(n-2)^2}{\lfloor 3} x^3 + \&c$$

$$\frac{1}{1+}\,\frac{x}{1+}\,\frac{x}{1+}\,\frac{2x}{1+}\cdots\cdots+\frac{(n-1)\,x}{1+}\,\frac{(n-1)\,x}{1}.$$

$$D = 1 + \frac{n^2}{\lfloor 1}\left(1-\frac{1}{n}\right)x + \frac{n^2(n-1)^2}{\lfloor 2}\left(1-\frac{2}{n}\right)x^2 + \frac{n^2(n-1)^2(n-2)^2}{\lfloor 3}\left(1-\frac{3}{n}\right)x^3 + \&c$$

If $\phi(a,n) = 1 + ax\cdot\frac{1-x^n}{1-x} + a^2x^4\cdot\frac{(1-x^{n-1})(1-x^{n-1})}{(1-x)(1-x^2)}$

$$+ a^3x^9\cdot\frac{(1-x^{n-2})(1-x^{n-3})(1-x^{n-4})}{(1-x)(1-x^2)(1-x^3)} + \&c,\ \text{then}$$

$$\frac{\phi(ax,n-1)}{\phi(a,n)} = \frac{1}{1+}\,\frac{ax}{1+}\,\frac{ax^2}{1+}\,\frac{ax^3}{1+}\cdots\cdots+\frac{ax^n}{1}.$$

If $\int_0^{\alpha} \frac{d\phi}{\sqrt{1-x\sin^2\phi}} = i\int_0^{\beta}\frac{d\phi}{\sqrt{1-(1-x)\sin^2\phi}}$

then $\alpha = \log\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right)$.

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vi. $\frac{\cos 2x}{2} - \frac{1}{2}\cdot\frac{\cos 4x}{4} + \frac{1.3}{2.4}\frac{\cos 6x}{6} - \&c = \cos\frac{x}{2}\sqrt{2\cos x} - 1$.

vii. $\frac{\sin x}{1} - \frac{1}{2}\cdot\frac{\sin 3x}{3} + \frac{1.3}{2.4}\cdot\frac{\sin 5x}{5} - \&c = \sin^{-1}(\sqrt{2}\sin\frac{x}{2})$.

viii. $\frac{\cos x}{1} - \frac{1}{2}\cdot\frac{\cos 3x}{3} + \frac{1.3}{2.4}\cdot\frac{\cos 5x}{5} - \&c = \log_e(\sqrt{\cos x} + \sqrt{2}\cos\frac{x}{2})$

ix. $\frac{\sin 2x}{2^2} - \frac{1}{2}\cdot\frac{\sin 4x}{4^2} + \frac{1.3}{2.4}\cdot\frac{\sin 6x}{6^2} - \&c = \sin\frac{x}{2}\sqrt{2\cos x}$
$+ \sin^{-1}(\sqrt{2}\sin\frac{x}{2}) - x$.

x. $\frac{\cos 2x}{2^2} - \frac{1}{2}\cdot\frac{\cos 4x}{4^2} + \frac{1.3}{2.4}\cdot\frac{\cos 6x}{6^2} - \&c = \cos\frac{x}{2}\sqrt{2\cos x}$
$- \log_e(\sqrt{\cos x} + \sqrt{2}\cos\frac{x}{2}) - 1 + \log_e 2$.

Similarly we can form similar identities for the II part also. From the additions of the I part the following theorem is obtained.

β. Let $F(n) = \left\{\frac{\sin x}{1^n} - \frac{1}{2}\cdot\frac{\sin 3x}{3^n} + \frac{1.3}{2.4}\cdot\frac{\sin 5x}{5^n} - \&c\right\}$
$- \cos\pi n\left\{\left(\frac{\sin 2x}{2^n} - \frac{1}{2}\cdot\frac{\sin 4x}{4^n} + \frac{1.3}{2.4}\cdot\frac{\sin 6x}{6^n} - \&c\right)\right.$
$\left. - \left(\frac{\sin 2x}{2^{n+1}} - \frac{1}{2}\cdot\frac{\sin 4x}{4^{n+1}} + \frac{1.3}{2.4}\cdot\frac{\sin 6x}{6^{n+1}} - \&c\right)\right\}$ and

$\psi(n) = \left\{\frac{\cos x}{1^n} - \frac{1}{2}\cdot\frac{\cos 3x}{3^n} + \frac{1.3}{2.4}\cdot\frac{\cos 5x}{5^n} - \&c\right\}$
$+ \cos\pi n\left\{\left(\frac{\cos 2x}{2^n} - \frac{1}{2}\cdot\frac{\cos 4x}{4^n} + \frac{1.3}{2.4}\cdot\frac{\cos 6x}{6^n} - \&c\right)\right.$
$\left. - \left(\frac{\cos 2x}{2^{n+1}} - \frac{1}{2}\cdot\frac{\cos 4x}{4^{n+1}} + \frac{1.3}{2.4}\cdot\frac{\cos 6x}{6^{n+1}} - \&c\right)\right\}$, then

If n is odd,

i. $\frac{F(n)}{2}\sin\frac{\pi n}{2} = \frac{x^n}{\lfloor n}S_0\phi(0) - \frac{x^{n-2}}{\lfloor n-2}\left\{S_0\phi(2) + \frac{S_2}{2^2}\phi(0)\right\}$
$+ \frac{x^{n-4}}{\lfloor n-4}\left\{S_0\phi(4) + \frac{S_2}{2^2}\phi(2) + \frac{S_4}{2^4}\phi(0)\right\} - \&c$
$= \frac{A_{n-1}}{\lfloor n-1}\phi(0) - \frac{A_{n-3}}{\lfloor n-3}\phi(2) + \frac{A_{n-5}}{\lfloor n-5}\phi(4) - \&c$ I

ii. $\frac{F(n+1)}{2}\sin\frac{\pi n}{2} = \frac{x^n}{\lfloor n}S_0\phi(1) - \frac{x^{n-2}}{\lfloor n-2}\left\{S_0\phi(3) + \frac{S_2}{2^2}\phi(1)\right\}$

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$$\frac{(1-e^{-x})}{1^2} + \frac{(1-e^{-2x})^2}{2^2} + \frac{(1-e^{-x})^3}{3^2} + \&c$$

$$= x - \frac{x^2}{4} + \beta_2 \frac{x^3}{\lfloor 3} - \beta_4 \frac{x^5}{\lfloor 5} + \beta_6 \frac{x^7}{\lfloor 7} - \&c$$

$$\begin{cases} S = \dfrac{\sin\frac{\theta}{2}}{\sinh\frac{y}{2}} + \dfrac{\sin\frac{3\theta}{2}}{\sinh\frac{3y}{2}} + \dfrac{\sin\frac{5\theta}{2}}{\sinh\frac{5y}{2}} + \&c \\ C = \dfrac{\cos\frac{\theta}{2}}{\cosh\frac{y}{2}} + \dfrac{\cos\frac{3\theta}{2}}{\cosh\frac{3y}{2}} + \dfrac{\cos\frac{5\theta}{2}}{\cosh\frac{5y}{2}} + \&c \\ C_1 = \frac{1}{2} + \dfrac{\cos\theta}{\cosh y} + \dfrac{\cos 2\theta}{\cosh 2y} + \dfrac{\cos 3\theta}{\cosh 3y} + \&c \end{cases}$$

$$CS = \frac{\sin\theta}{\cosh y} + \frac{2\sin 2\theta}{\cosh 2y} + \frac{3\sin 3\theta}{\cosh 3y} + \&c$$

$$\therefore CS + \frac{dC_1}{d\theta} = 0 \quad \therefore C_1 S + \frac{dC}{d\theta} = 0 \quad \& \quad CC_1 = \frac{dS}{d\theta}.$$

$$\therefore C^2 + S^2 = x\frac{x^2}{4} \quad \& \quad C_1^2 + S^2 = \frac{x^2}{4}.$$

$$\therefore \text{Let } C = \sqrt{x}\cdot\frac{x}{2}\cos\phi \quad \& \quad S = \sqrt{x}\cdot\frac{x}{2}\sin\phi.$$

$$\therefore C_1 = \frac{x}{2}\sqrt{1 - x\sin^2\phi}$$

$$\therefore \frac{x}{2}\cos\phi\sqrt{1 - x\sin^2\phi} = \frac{d\sin\phi}{d\theta} = \cos\phi\frac{d\phi}{d\theta}.$$

$$\therefore \theta = \frac{2}{x}\int_0^{\phi} \frac{d\phi}{\sqrt{1 - x\sin^2\phi}}.$$

$$\begin{cases} \sin\phi \text{ ~ } i\tan\phi \\ \cos\phi \text{ ~ } \sec\phi \\ \phi \text{ ~ } i\log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \end{cases} \qquad \begin{cases} \theta \text{ ~ } i\theta\frac{x}{x'} \\ y \text{ ~ } y' \\ x \text{ ~ } 1-x \\ x \text{ ~ } x' \end{cases}$$

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$$+\frac{x^{n-4}}{\lfloor n-4}\left\{S_0\,\phi(5)+\frac{S_2}{2^2}\phi(3)+\frac{S_4}{4}\phi(1)\right\}-\&c$$

$$=\frac{A_{n-1}}{\lfloor n-1}\phi(1)-\frac{A_{n-3}}{\lfloor n-3}\phi(3)+\frac{A_{n-5}}{\lfloor n-5}\phi(5)-\&c \qquad \text{II}$$

where $S_n=\frac{1}{1^n}-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\&c$

$$\frac{\pi}{2}\phi(n)=\frac{1}{1^{n+1}}+\frac{1}{2}\cdot\frac{1}{3^{n+1}}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5^{n+1}}+\&c$$

and $\frac{2}{\pi}A_n=\left(\frac{\pi}{2}\right)^n+\left(\frac{3\pi}{2}\right)^n+\left(\frac{5\pi}{2}\right)^n+\cdots\cdots+\left(x-\frac{\pi}{2}\right)^n$.

If n is even $\frac{\psi(n)}{2}\cos\frac{\pi n}{2}=\text{I}$ & $\frac{\psi(n+1)}{2}\cos\frac{\pi n}{2}=\text{II}$.

8. If $\phi(x)=\frac{x}{1^2}-\frac{x^2}{2^2}+\frac{x^3}{3^2}-\frac{x^4}{4^2}+\&c$, then

i. $\phi(x-1)+\phi\left(\frac{1}{x}-1\right)=\frac{1}{2}(\log_e x)^2$.

ii. $\phi(x)+\phi\left(\frac{1}{x}\right)=\frac{1}{2}(\log_e x)^2+\frac{\pi^2}{6}$

iii. $\phi(-x)+\phi(x-1)=\log_e x\,\log_e(1-x)-\frac{\pi^2}{6}$

iv. $\phi(x)+\phi(-x)=\frac{1}{2}\phi(x^2)$.

v. $\phi(x)-\phi(-x)=\psi(x)=2\left(\frac{x}{1^2}+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\&c\right)$

$\psi(x)+\psi\left(\frac{1-x}{1+x}\right)=\frac{\pi^2}{4}+\log_e x\,\log_e\frac{1+x}{1-x}$.

Ex. 1. $\frac{1}{1^2}\cdot\frac{1}{2}+\frac{1}{2^2}\cdot\frac{1}{2^2}+\frac{1}{3^2}\cdot\frac{1}{2^3}+\frac{1}{4^2}\cdot\frac{1}{2^4}+\&c=\frac{\pi^2}{12}-\frac{1}{2}(\log_e 2)^2$.

2. $\frac{1}{1^2}\cdot\left(\frac{\sqrt5-1}{2}\right)+\frac{1}{2^2}\cdot\left(\frac{\sqrt5-1}{2}\right)^2+\frac{1}{3^2}\cdot\left(\frac{\sqrt5-1}{2}\right)^3+\&c=\frac{\pi^2}{10}-\left(\log_e\frac{\sqrt5+1}{2}\right)^2$

3. $\frac{1}{1^2}\cdot\left(\frac{3-\sqrt5}{2}\right)+\frac{1}{2^2}\cdot\left(\frac{3-\sqrt5}{2}\right)^2+\frac{1}{3^2}\cdot\left(\frac{3-\sqrt5}{2}\right)^3+\&c=\frac{\pi^2}{15}-\left(\log_e\frac{\sqrt5+1}{2}\right)^2$.

4. $\frac{\sqrt2-1}{1^2}+\frac{(\sqrt2-1)^3}{3^2}+\frac{(\sqrt2-1)^5}{5^2}+\&c=\frac{\pi^2}{16}-\frac{1}{4}\{\log_e(1+\sqrt2)\}^2$

5. $\frac{1}{1^2}\cdot\left(\frac{\sqrt5-1}{2}\right)+\frac{1}{3^2}\cdot\left(\frac{\sqrt5-1}{2}\right)^3+\frac{1}{5^2}\cdot\left(\frac{\sqrt5-1}{2}\right)^5+\&c=\frac{\pi^2}{12}-\frac{3}{4}\left(\log_e\frac{\sqrt5+1}{2}\right)^2$

6. $\frac{\sqrt5-2}{1^2}+\frac{(\sqrt5-2)^3}{3^2}+\frac{(\sqrt5-2)^5}{5^2}+\&c=\frac{\pi^2}{24}-\frac{3}{4}\left(\log_e\frac{\sqrt5+1}{2}\right)^2$

9. If $\phi(x)=\frac{x}{1^3}-\frac{x^2}{2^3}+\frac{x^3}{3^3}-\frac{x^4}{4^3}+\&c$, then

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If $F(\alpha,\beta) = \frac{\alpha}{n+} \frac{(\beta)^2}{n+} \frac{(2\alpha)^2}{n+} \frac{(3\beta)^2}{n+} \&c$

~~then F α, β~~ & A, G the A.M & G.M. between α & β

then $F(A, G)$ is the A.M between $F(\alpha,\beta)$ & $F(\beta,\alpha)$

If $a_1 + a_2 = b_1 + b_2 = c_1 + c_2 = \&c = p$

$a_2 a_3 = b_2 b_3 = c_2 c_3 = d_2 d_3 \&c = q$

$a_3 + a_4 = b_3 + b_4 = c_3 + c_4 = \&c = r$

$a_4 a_5 = b_4 b_5 = c_4 c_5 = \&c = s$ &c &c

then $\Sigma \frac{a}{1+} \frac{a_1}{1+} \frac{a_2}{1+} \frac{a_3}{1+} \&c$

$= \Sigma a - \frac{\Sigma a a_1}{1+p-} \frac{q}{1+r-} \frac{s}{1+t-} \frac{u}{1+v-} \&c$

If $F(\alpha,\beta) = \tan^{-1} \frac{\alpha}{x+} \frac{\beta^2+K^2}{x+} \frac{\alpha^2+(2K)^2}{x+} \frac{\beta^2+(3K)^2}{x+\&c.}$

and A the A.M between α & β, then

$F(A, A)$ is the A.M between $F(\alpha,\beta)$ & $F(\beta,\alpha)$.

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i. $\phi(x-1) + \phi(\frac{1}{x}-1) + \phi(-x) = \frac{1}{2}(\log_e x)^2 \log_e(1-x) - \frac{1}{6}(\log_e x)^3 - \frac{\pi^2}{6}\log_e x - S_3$.

ii. $\phi(x) - \phi(\frac{1}{x}) = \frac{1}{6}(\log_e x)^3 + \frac{\pi^2}{6}\log_e x$.

iii. $\phi(x) + \phi(-x) = \frac{1}{4}\phi(-x^4)$.

Ex. 1. $\frac{1}{1^3}\cdot\frac{1}{2} + \frac{1}{2^3}\cdot\frac{1}{2^2} + \frac{1}{3^3}\cdot\frac{1}{2^3} + \frac{1}{4^3}\cdot\frac{1}{2^4} + \&c = \frac{1}{6}(\log_e 2)^3 - \frac{\pi^2}{12}\log_e 2 + (\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c)$.

2. $\frac{1}{1^3}\cdot\frac{3-\sqrt{5}}{2} + \frac{1}{2^3}\cdot\left(\frac{3-\sqrt{5}}{2}\right)^2 + \frac{1}{3^3}\cdot\left(\frac{3-\sqrt{5}}{2}\right)^3 + \&c = \frac{2}{3}\left(\log_e \frac{\sqrt{5}+1}{2}\right)^3 - \frac{2\pi^2}{15}\log_e \frac{\sqrt{5}+1}{2} + S_3$

10. If $\phi(x) = x + (1+\frac{1}{3})\frac{x^3}{3} + (1+\frac{1}{3}+\frac{1}{5})\frac{x^5}{5} + \&c$

then $\phi\left(\frac{x}{2-x}\right) = \frac{1}{8}\left\{\log_e(1-x)\right\}^2 + \frac{1}{2}\left(\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \&c\right)$

Ex 1. $\frac{1}{3} + \frac{1+\frac{1}{3}}{3}\cdot\frac{1}{3^3} + \frac{1+\frac{1}{3}+\frac{1}{5}}{5}\cdot\frac{1}{3^5} + \&c = \frac{\pi^2}{24} - \frac{1}{8}(\log_e 2)^2$.

2. $\frac{1}{5} + \frac{1+\frac{1}{3}}{3}\cdot\frac{1}{5^3} + \frac{1+\frac{1}{3}+\frac{1}{5}}{5}\cdot\frac{1}{5^5} + \&c = \frac{\pi^2}{20\sqrt{5}} - \frac{3}{8\sqrt{5}}\left(\log_e \frac{\sqrt{5}+1}{2}\right)^2$

3. $(\sqrt{5}-2) + \frac{1+\frac{1}{3}}{3}(\sqrt{5}-2)^3 + \frac{1+\frac{1}{3}+\frac{1}{5}}{5}(\sqrt{5}-2)^5 + \&c = \frac{\pi^2}{30} - \frac{3}{4}\left(\log_e \frac{\sqrt{5}+1}{2}\right)^2$.

11. If $\phi(x) = \frac{x^2}{2^2} + (1+\frac{1}{2})\frac{x^3}{3^2} + (1+\frac{1}{2}+\frac{1}{3})\frac{x^4}{4^2} + \&c$, then

i. $\phi(1-x) = \frac{1}{2}\log_e(1-x)(\log_e x)^2 + \log_e x\left(\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \&c\right) - \left(\frac{x}{1^3} + \frac{x^2}{2^3} + \frac{x^3}{3^3} + \&c\right) + S_3$

ii. $\phi(1-x) - \phi(1-\frac{1}{x}) = \frac{1}{6}(\log_e x)^3$

iii. $\phi(1-x) = \frac{1}{2}\log_e(1-x)(\log_e x)^2 - \frac{1}{3}(\log_e x)^3 - \log_e x\left(\frac{1}{1^2 x} + \frac{1}{2^2 x^2} + \&c\right) - \left(\frac{1}{1^3 x} + \frac{1}{2^3 x^2} + \frac{1}{3^3 x^3} + \&c\right) + S_3$.

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$$\int_0^\infty e^{-x}\left(1+\frac{x}{n}\right)^n dx = 1 + \cfrac{n}{1 + \cfrac{1(n-1)}{3 + \cfrac{2(n-2)}{5 + \cfrac{3(n-3)}{7 + \&c}}}}$$

$$= 2 + \cfrac{n-1}{2 + \cfrac{1(n-2)}{4 + \cfrac{2(n-3)}{6 + \cfrac{3(n-4)}{8 + \&c}}}}$$

$$= \frac{e^n \lfloor n}{n^n} - \cfrac{2n}{2 + \cfrac{3n}{3 + \cfrac{4n}{4 + \cfrac{5n}{5 +}}}}$$

$$\frac{\operatorname{cosech}\frac{y}{2}}{1+n^2} - \frac{\operatorname{cosech}\frac{3y}{2}}{1+(3n)^2} + \frac{\operatorname{cosech}\frac{5y}{2}}{1+(5n)^2} - \&c$$

$$= \cfrac{\frac{z}{2}\sqrt{x}}{1 + \cfrac{(1-x)(nz)^2}{1 - \cfrac{x(2nz)^2}{1 + \cfrac{(1-x)(3nz)^2}{1 - \&c}}}}$$

$$\int_0^\infty e^{-n\int_0^\phi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}} \frac{\cos\phi}{1-x\sin^2\phi}\, d\phi$$

$$\underset{R}{\mathfrak{R}}\,\underset{R}{R}\,\mathfrak{R} = \cfrac{1}{n + \cfrac{1-x}{n - \cfrac{4x}{n + \cfrac{9(1-x)}{n - \cfrac{16x}{n + \&c}}}}}$$

If $1+\sqrt{1-\alpha} = \dfrac{2}{1+\sqrt{\beta}} = \gamma$

then $\cfrac{\sqrt{\beta}}{n + \cfrac{1^2}{n + \cfrac{2^2\beta}{n + \cfrac{3^2}{n + \cfrac{4^2\beta}{n + \cfrac{5^2}{n + \&c}}}}}}$

$$+ \cfrac{1}{n + \cfrac{1^2\beta}{n + \cfrac{2^2}{n + \cfrac{3^2\beta}{n + \cfrac{4^2}{n + \&c}}}}}$$

$$= \cfrac{2}{n\gamma + \cfrac{1^2\alpha}{n\gamma + \cfrac{2^2}{n\gamma + \cfrac{3^2\alpha}{n\gamma + \cfrac{4^2}{n\gamma + \&c}}}}}$$

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IV. $\phi(-x) + \phi(-\frac{1}{x}) = -\frac{1}{6}(\log_e x)^3 + \log_e x\left(\frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c\right)$
$-\left(\frac{x}{1^3} - \frac{x^2}{2^3} + \frac{x^3}{3^3} - \&c\right) + S_3$

12. If $\phi(x) = \frac{x^2}{2^3} - (1+\frac{1}{2})\frac{x^3}{3^3} + (1+\frac{1}{2}+\frac{1}{3})\frac{x^4}{4^3} - \&c$, then

i. $\phi(x-1) - \phi(\frac{1}{x}-1) = -\frac{1}{24}(\log_e x)^4 + \frac{1}{6}(\log_e x)^3 \log_e(1-x)$
$+ \log_e x\left(\frac{x}{1^3} + \frac{x^2}{2^3} + \frac{x^3}{3^3} + \&c\right) - 2\left(\frac{x}{1^4} + \frac{x^2}{2^4} + \frac{x^3}{3^4} + \&c\right)$
$+ S_3 \log_e x + \frac{\pi^4}{45}$.

ii. $\phi(x) - \phi(\frac{1}{x}) = -\frac{1}{24}(\log_e x)^4 + \log_e x\left(\frac{x}{1^3} - \frac{x^2}{2^3} + \frac{x^3}{3^3} - \&c\right)$
$-2\left(\frac{x}{1^4} - \frac{x^2}{2^4} + \frac{x^3}{3^4} - \&c\right) + S_3 \log_e x + \frac{7\pi^4}{360}$.

13. If $\phi(x) = \frac{x^2}{2^2} + (1+\frac{1}{3})\frac{x^4}{4^2} + (1+\frac{1}{3}+\frac{1}{5})\frac{x^6}{6^2} + \&c$ and
$\psi(x) = \frac{x^2}{2^3} + (1+\frac{1}{3})\frac{x^4}{4^3} + (1+\frac{1}{3}+\frac{1}{5})\frac{x^6}{6^3} + \&c$ then

i. $\phi\left(\frac{1-x}{1+x}\right) = \frac{1}{8}(\log_e x)^2 \log_e \frac{1-x}{1+x} + \frac{1}{2}\log_e x\left(\frac{x}{1^2} + \frac{x^3}{3^2} + \frac{x^5}{5^2} + \&c\right)$
$+ \frac{1}{2}\left(\frac{1-x}{1^3} + \frac{1-x^3}{3^3} + \frac{1-x^5}{5^3} + \&c\right)$

ii. $\psi(x) + \psi\left(\frac{1-x}{1+x}\right) = \phi(x)\log_e x + \phi\left(\frac{1-x}{1+x}\right)\log_e \frac{1-x}{1+x}$
$-\frac{1}{16}(\log_e x)^2\left(\log_e \frac{1-x}{1+x}\right)^2$.
$+ \frac{\pi}{4}\left(\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \&c\right) - \frac{\pi}{3\sqrt{3}}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$.

14. If $\phi(x) = x + (1+\frac{1}{2})\frac{x^3}{3} + (1+\frac{1}{2}+\frac{1}{3})\frac{x^5}{5} + \&c$ then

$\phi\left(\frac{1-x}{1+x}\right) = -(1-\log_e 2)\log_e x + \frac{1+x}{1-x}\log_e \frac{4x}{(1+x)^2} + \frac{1}{4}(\log_e x)^2$
$-\left(\frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c\right) + \frac{\pi^2}{12}$.

Ex. 1. $\frac{1}{2} + \frac{1+\frac{1}{2}}{2^2}\cdot\frac{1}{2^2} + \frac{1+\frac{1}{2}+\frac{1}{3}}{3^2}\cdot\frac{1}{2^3} + \&c = S_3 - \frac{\pi^2}{12}\log 2$

2. $\frac{1}{1^2} + \frac{1+\frac{1}{2}+\frac{1}{3}}{3^2} + \frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}}{5^2} + \&c = \frac{3}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$

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If $I(n)$ be the nearest integer to

$$\frac{1}{4n}\left\{\cosh \pi\sqrt{n} - \frac{\sinh \pi\sqrt{n}}{\pi\sqrt{n}}\right\}$$

then $I(0) + xI(1) + x^2 I(2) + x^3 I(3) + \&c$

$$= \frac{1}{1 - 2x + 2x^4 - 2x^9 + 2x^{16} - \&c}.$$

If $\int_0^\infty \frac{e^{-m^2x^2}}{1+x^2}\,dx = \phi(m)$, & if $m \neq n$.

then $\int_0^\infty \frac{e^{-m^2x^2}}{1+x^2}\cos 2mnx\,dx = \frac{1}{2}e^{-n^2}\{\phi(m+n) + \phi(m-n)\}$

$$\int_0^\infty e^{-x}\left(1+\frac{x}{n}\right)^n dx = \frac{e^n \lfloor n}{2n^n} +$$

$$\frac{2}{3} - \frac{4}{135n} + \frac{8}{27.105\,n^2} + \frac{16}{105.81\,n^3} + \&c$$

$$(m-n-1)\int_0^\infty \frac{\left(1+\frac{x}{n}\right)^n}{\left(1+\frac{x}{m}\right)^m} dx = \frac{m}{2}\cdot\frac{m^m \lfloor n}{n^n \lfloor m}\cdot\frac{\lfloor m-n}{(m-n)^{m-n}}$$

$$+ \frac{2}{3}(m+n) - \frac{4(m+n)(m-2n)\left(m-\frac{n}{2}\right)}{135mn(m-n)}$$

$$+ \frac{8(m^3+n^3)(m-2n)\left(m-\frac{n}{2}\right)}{27.105\,m^2n^2(m-n)^2} + \frac{16(m^3+n^3)}{105.81\,m^3n^3}$$

$$- \frac{32.281}{3^8.5^2.7.11\,n^4} - \&c \times \frac{(m^2-mn+n^2)(m-2n)}{(m-n)^3}\times\left(m-\frac{n}{2}\right)$$

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3. $\frac{1}{1^2} + \frac{1+\frac{1}{3}}{2^2} + \frac{1+\frac{1}{3}+\frac{1}{5}}{3^2} + \&c = 2\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$

4. $(\sqrt{5}-2) + \frac{1+\frac{1}{2}}{3}(\sqrt{5}-2)^3 + \frac{1+\frac{1}{2}+\frac{1}{3}}{5}(\sqrt{5}-2)^5 + \&c$

$= \frac{\pi^2}{60} + \frac{3}{4}\left(\log_e \frac{\sqrt{5}+1}{2}\right)^2 + (\sqrt{5}+2)\log_e 4 + (3\sqrt{5}+5+\log_e 2)\log_e \frac{\sqrt{5}-1}{2}$.

15. $S_{n+1}\cos\frac{\pi n}{2}\lfloor n = \int \frac{x^n}{2}\cot\frac{x}{2}\,dx +$

$x^n\left(\frac{\cos x}{1} + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \&c\right)$

$- n x^{n-1}\left(\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \&c\right)$

$- n(n-1)x^{n-2}\left(\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \&c\right)$

$+ n(n-1)(n-2)x^{n-3}\left(\frac{\sin x}{1^3} + \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} + \&c\right)$

$+ \&c$ where $S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c$ & $S_1 = \log_e 2$.

Sol. $\sin x + \sin 2x + \sin 3x + \&c = \frac{1}{2}\cot\frac{x}{2}$

$\therefore \int x^n(\sin x + \sin 2x + \&c)\,dx = \int \frac{x^n}{2}\cot\frac{x}{2}\,dx$

and apply $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \&c$.

16. $S_{n+1}\cos\frac{\pi n}{2}\lfloor n = \int \frac{x^n}{2\sin x}\,dx$

$+ x^n\left(\frac{\cos x}{1} + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \&c\right)$

$- n x^{n-1}\left(\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \&c\right)$

$- n(n-1)x^{n-2}\left(\frac{\cos x}{1^3} + \frac{\cos 3x}{3^3} + \frac{\cos 5x}{5^3} + \&c\right) + \&c$

Sol. $\sin x + \sin 3x + \sin 5x + \&c = \frac{1}{2}\operatorname{Cosec} x$.

17. If $\int x^n \cot x\,dx = f_n(x)$, then

$2^n f_n\left(\frac{\pi}{2} - x\right) = \pi^n\{f_0(2x) - f_0(x)\} - \frac{n}{\lfloor 1}\pi^{n-1}\{f_1(2x) - 2f_1(x)\}$

$+ \frac{n(n-1)}{\lfloor 2}\pi^{n-2}\{f_2(2x) - 2^2 f_2(x)\} - \frac{n(n-1)(n-2)}{\lfloor 3}\pi^{n-3}$

$\{f_3(2x) - 2^3 f_3(x)\}$ +

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$$\frac{\operatorname{Sech}\frac{y}{2}}{1+x^2}+\frac{\operatorname{Sech}\frac{3y}{2}}{1+(3x)^2}+\frac{\operatorname{Sech}\frac{5y}{2}}{1+(5x)^2}+\&c$$

$$=\frac{1}{2}\cdot\frac{z\sqrt{x}}{1+\dfrac{(xz)^2}{1+\dfrac{(2xz)^2x}{1+\dfrac{(3xz)^2}{1+\dfrac{(4xz)^2x}{1+\&c}}}}}$$

$$\int_0^\infty e^{-x}\int_0^{\phi}\frac{d\theta}{\sqrt{1-x\sin^2\theta}}\cdot\frac{\cos\phi}{\sqrt{1-x\sin^2\phi}}\,d\phi=\frac{1}{x+\dfrac{1}{x+\dfrac{4x}{x+\frac{9}{x}}}}+\frac{16x}{x+}\&c$$

$$\frac{\operatorname{Sech}\frac{y}{2}}{1+x^2}-\frac{3\operatorname{Sech}\frac{3y}{2}}{1+(3x)^2}+\frac{5\operatorname{Sech}\frac{5y}{2}}{1+(5x)^2}-\&c.$$

$$=\frac{1}{2}\cdot\frac{z^2\sqrt{x(1-x)}}{1+x^2z^2(1-2x)}+\frac{2^2(2^2-1)x(1-x)x^4z^4}{1+(3xz)^2(1-2x)}+$$

$$\frac{4^2(4^2-1)x(1-x)x^4z^4}{1+(5xz)^2(1-2x)}+\&c$$

$$\frac{\operatorname{Cosech}\frac{y}{2}}{1+x^2}+\frac{3\operatorname{Cosech}\frac{3y}{2}}{1+(3x)^2}+\frac{5\operatorname{Cosech}\frac{5y}{2}}{1+(5x)^2}+\&c$$

$$=\frac{1}{2}\cdot\frac{z^2\sqrt{x}}{1+(xz)^2(1+x)}-\frac{2^2(2^2-1)xx^4z^4}{1+(3xz)^2(1+x)}-$$

$$\frac{4^2(4^2-1)xx^4z^4}{1+(5xz)^2(1+x)}-\&c$$

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Sol. $\tan x = \cot x - 2\cot 2x$ and

$$f_n\left(\frac{\pi}{2}-x\right) = \int \left(\frac{\pi}{2}-x\right)^n \cot\left(\frac{\pi}{2}-x\right) d\left(\frac{\pi}{2}-x\right) = -\int \left(\frac{\pi}{2}-x\right)^n \tan x\, dx$$

N. B. Let $\sin x = y$ & $\tan x = z$, then

$$\int x^n \cot x\, dx = \int \frac{x^n}{\sin x}\cos x\, dx = \int \frac{(\sin^{-1}y)^n}{y}\, dy \text{ and}$$

$$\int \frac{2x^n}{\sin 2x}\, dx = \int \frac{x^n}{\cos x \sin x}\, dx = \int \frac{x^n}{\tan x}\sec^2 x\, dx$$

$$= \int \frac{(\tan^{-1}z)^n}{z}\, dz.$$ So we have to find $(\sin^{-1}y)^n$ & $(\tan^{-1}y)^n$.

i. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \&c$

ii. $\frac{1}{2}(\tan^{-1}x)^2 = \frac{x^2}{2} - \left(+\frac{1}{3}\right)\frac{x^4}{4} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{x^6}{6} - \&c$

iii. $\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \&c$

iv. $\frac{1}{2}(\sin^{-1}x)^2 = \frac{x^2}{2} + \frac{1}{3}\cdot\frac{x^4}{4} + \frac{1.4}{3.5}\cdot\frac{x^6}{6} + \frac{2.4.6}{3.5.7}\cdot\frac{x^8}{8} + \&c$

v. $\frac{1}{6}(\sin^{-1}x)^3 = \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5}\left(1+\frac{4}{32}\right) + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}\left(1+\frac{7}{32}+\frac{54}{}\right)$

$+ \&c$

vi. $\frac{1}{24}(\sin^{-1}x)^4 = \frac{2}{3}\cdot\frac{x^4}{4}\cdot\frac{1}{2} + \frac{2.4}{3.5}\cdot\frac{x^6}{8}\cdot\left(\frac{1}{2}+\frac{1}{42}\right) + \&c$

vii $\frac{d(\sin^{-1}x)^n}{dx} = n\frac{(\sin^{-1}x)^{n-1}}{\sqrt{1-x^2}}$ & $\frac{d(\tan^{-1}x)^n}{dx} = n\cdot\frac{(\tan^{-1}x)^{n-1}}{1+x^2}$.

18. $\frac{\sin x}{1^2} + \frac{1}{2}\cdot\frac{\sin^3 x}{3^2} + \frac{1.3}{2.4}\cdot\frac{\sin^5 x}{5^2} + \frac{1.3.5}{2.4.6}\cdot\frac{\sin^7 x}{7^2} + \&c$

$= x\log_e 2\sin x + \frac{1}{2}\left(\frac{\sin 2x}{1^2} + \frac{\sin 4x}{2^2} + \frac{\sin 6x}{3^2} + \&c\right)$

Ex. 1. $\frac{1}{1^2} + \frac{1}{2}\cdot\frac{1}{3^2} + \frac{1.3}{2.4}\cdot\frac{1}{5^2} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{7^2} + \&c = \frac{\pi}{2}\log_e 2.$

2. $\frac{1}{1^2} + \frac{1}{2}\cdot\frac{1}{3^2}\cdot\frac{1}{2} + \frac{1.3}{2.4}\cdot\frac{1}{5^2}\cdot\frac{1}{2^2} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{7^2}\cdot\frac{1}{2^3} + \&c$

$= \frac{\pi}{4\sqrt{2}}\log_e 2 + \frac{1}{\sqrt{2}}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c\right)$

3. $\frac{1}{1^2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3^2}\cdot\frac{1}{2^3} + \frac{1.3}{2.4}\cdot\frac{1}{5^2}\cdot\frac{1}{2^5} + \&c$

$= \frac{3\sqrt{3}}{4}\left(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \&c\right) - \frac{\pi^2}{6\sqrt{3}}$

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If $\alpha\beta = 4\pi^2$ & $F(x) = \frac{\sqrt{5}+1}{2} + \frac{\sqrt[5]{x}}{1+\frac{x}{1+\frac{x^2}{1+\frac{x^3}{1+\&c}}}}$

then $F(e^{-\alpha})\, F(e^{-\beta}) = \frac{5+\sqrt{5}}{2}$.

$f^2(-x^2,-x^3) - \sqrt[5]{x^2} f^2(-x,-x^4)$

$= f(-x,-x^4)\left\{ f(-\sqrt[5]{x},-\sqrt[5]{x^2}) + \sqrt[5]{x} f(-x^5,-x^{10})\right\}$.

If $K = \frac{f(-x^{\frac{1}{5}},-x^{\frac{2}{5}})}{\sqrt[5]{x} f(-x^5,-x^{10})}$ then

$$\frac{\sqrt{K^2+2K+5} - (K+1)}{2} = \frac{\sqrt[5]{x}}{1+\frac{x}{1+\frac{x^2}{1+\frac{x^3}{1+\&c}}}}$$

If $\int_0^\infty \phi(x) \cos 2\pi x\, dx = \frac{\sqrt{\pi}}{2}\psi(x)$

then $\int_0^\infty e^{-x^2}\phi(x)\,dx = \int_0^\infty e^{-x^2}\psi(x)\,dx$

&c &c &c

$$\int_0^\infty \frac{F(a+bx) - F(a-bx)}{\sinh}$$

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4. $\frac{1}{1^2} + \frac{1}{2}\cdot\frac{1}{3^2}\cdot\left(\frac{3}{4}\right) + \frac{1.3}{2.4}\cdot\frac{1}{5^2}\cdot\left(\frac{3}{4}\right)^2 + \frac{1.3.5}{2.4.6}\cdot\frac{1}{7^2}\cdot\left(\frac{3}{4}\right)^3 + \&c.$

$= \frac{\pi}{3\sqrt{3}}\log_e 3 - \frac{2\pi^2}{27} + \left(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \&c\right).$

19. $\frac{\tan x}{1^2} - \frac{\tan^3 x}{3^2} + \frac{\tan^5 x}{5^2} - \frac{\tan^7 x}{7^2} + \&c$

$= x\log_e \tan x + \left(\frac{\sin 2x}{1^2} + \frac{\sin 6x}{3^2} + \frac{\sin 10x}{5^2} + \&c\right)$

Ex. 1. $\frac{1}{1^2}\cdot\frac{1}{3} - \frac{1}{3^2}\cdot\frac{1}{3^2} + \frac{1}{5^2}\cdot\frac{1}{3^3} - \&c$

$= -\frac{\pi}{12\sqrt{3}}\log_e 3 - \frac{5}{5.4}\pi^2 + \frac{5}{4}\left(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \&c\right)$

2. $\frac{\sqrt{2}-1}{1^2} - \frac{(\sqrt{2}-1)^3}{3^2} + \frac{(\sqrt{2}-1)^5}{5^2} - \frac{(\sqrt{2}-1)^7}{7^2} + \&c$

$= -\frac{\pi}{8}\log_e(1+\sqrt{2}) - \frac{\pi^2}{16} + \sqrt{2}\left(\frac{1}{1^2} - \frac{1}{5^2} + \frac{1}{9^2} - \frac{1}{13^2} + \&c\right)$

3. $\frac{2-\sqrt{3}}{1^2} - \frac{(2-\sqrt{3})^3}{3^2} + \frac{(2-\sqrt{3})^5}{5^2} - \&c = -\frac{\pi}{12}\log_e(2+\sqrt{3})$

$+ \frac{2}{3}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c\right)$

20. If x lies between 0 & $\frac{\pi}{4}$

$\frac{\cos x - \sin x}{1^2} + \frac{1}{2}\cdot\frac{\cos^3 x - \sin^3 x}{3^2} + \frac{1.3}{2.4}\cdot\frac{\cos^5 x - \sin^5 x}{5^2} + \&c$

$+ \frac{1}{2}\left\{\frac{\sin 2x}{1^2} + \frac{1}{2}\cdot\frac{\sin^3 2x}{3^2} + \frac{1.3}{2.4}\cdot\frac{\sin^5 2x}{5^2} + \&c\right\}$

$= \frac{\pi}{2}\log_e 2\cos x.$

Ex. If $\psi(x) = \frac{x}{1^2} + \frac{1}{2}\cdot\frac{x^3}{3^2} + \frac{1.3}{2.4}\cdot\frac{x^5}{5^2} + \&c$

shew that $\psi\left(\frac{3}{5}\right) - \frac{1}{2}\psi\left(\frac{24}{25}\right) = \frac{\pi}{2}\log_e 2 + 2\psi\left(\frac{1}{\sqrt{3}}\right) - 2\psi(2$

21. $\frac{\cos x + \sin x}{1^2} + \frac{1}{2}\cdot\frac{\cos^3 x + \sin^3 x}{3^2} + \frac{1.3}{2.4}\cdot\frac{\cos^5 x + \sin^5 x}{5^2} + \&$

$= \frac{\pi}{2}\log_e 2\cos x + \frac{\tan x}{1^2} - \frac{\tan^3 x}{3^2} + \frac{\tan^5 x}{5^2} - \&c.$

Ex. $\frac{1}{1^2}\cdot\frac{2+1}{5} + \frac{1}{2}\cdot\frac{1}{3^2}\cdot\frac{2^3+1}{5^2} + \frac{1.3}{2.4}\cdot\frac{1}{5^2}\cdot\frac{2^5+1}{5^3} + \&c$

$= \frac{\pi}{2\sqrt{5}}\log_e\frac{4}{\sqrt{5}} + \frac{1}{\sqrt{5}}\left(\frac{1}{1^2}\cdot\frac{1}{2} - \frac{1}{3^2}\cdot\frac{1}{2^3} + \frac{1}{5^2}\cdot\frac{1}{2^5} - \&c\right)$

22. $\frac{\sin^2 x}{2^2} + \frac{1}{3}\cdot\frac{\sin^4 x}{4^2} + \frac{1.4}{3.5}\cdot\frac{\sin^6 x}{6^2} + \&c.$

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$$x + \frac{x^2+1}{2x} + \frac{x^2+9}{2x} + \frac{x^2+25}{2x} + \&c$$

$$= x + \frac{x^2-1}{2x} + \frac{x^2-9}{2x} + \frac{x^2-25}{2x} + \&c \times \frac{1-e^{-\pi x}}{1-2e^{-\frac{\pi x}{2}}\sin\frac{\pi x}{2} + e^{-\pi x}}$$

$$\frac{1 + \frac{ax^2}{1-x} + \frac{a^2x^6}{(1-x)(1-x^2)} + \frac{a^3x^{12}}{(1-x)(1-x^2)(1-x^3)} + \&c}{1 + \frac{ax}{1-x} + \frac{a^2x^4}{(1-x)(1-x^2)} + \frac{a^3x^9}{(1-x)(1-x^2)(1-x^3)} + \&c}$$

$$= \frac{1}{1 + \frac{ax}{1 + \frac{ax^2}{1 + \frac{ax^3}{1 + \frac{ax^4}{1 + \&c}}}}}$$

$$\frac{1}{(1-ax)(1-ax^2)(1-ax^3)(1-ax^4)\,\&c}$$

$$= 1 + \frac{ax}{(1-x)(1-ax)} + \frac{a^2x^4}{(1-x)(1-x^2)(1-ax)(1-ax^2)} + \frac{a^3x^9}{(1-x)(1-x^2)(1-x^3)(1-ax)(1-ax^2)(1-ax^3)} + \&c$$

$$\frac{f(-x^5, -x^{10})}{f(-x, -x^4)} = 1 + \frac{x}{1-x} + \frac{x^4}{(1-x)(1-x^2)} + \frac{x^9}{(1-x)(1-x^2)(1-x^3)} + \&c$$

$$\frac{f(-x^5, -x^{10})}{f(-x^2, -x^3)} = 1 + \frac{x^2}{1-x} + \frac{x^6}{(1-x)(1-x^2)} + \frac{x^{12}}{(1-x)(1-x^2)(1-x^3)} + \&c.$$

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$= \frac{1}{2}x^2 \log_e 2\sin x + \frac{x}{2}\left(\frac{\sin 2x}{1^2} + \frac{\sin 4x}{2^2} + \frac{\sin 6x}{3^2} + \&c\right)$

$+ \frac{1}{4}\left(\frac{\cos 2x}{1^3} + \frac{\cos 4x}{2^3} + \frac{\cos 6x}{3^3} + \&c\right)$

Ex. 1. $\frac{1}{1^2} + \frac{2}{3}\cdot\frac{1}{4^2} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{6^2} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{8^2} + \&c$

$= \frac{\pi^2}{8}\log_e 2 - \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$

2. $\frac{1}{1^2}\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{1}{4^2}\cdot\frac{1}{2^2} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{6^2}\cdot\frac{1}{2^2} + \&c$

$= \frac{\pi^2}{64}\log_e 2 + \frac{\pi}{8}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c\right) - \frac{5}{16}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$

23. $\frac{\tan^2 x}{2^2} - \left(1 + \frac{1}{3}\right)\frac{\tan^4 x}{4^2} + \left(1 + \frac{1}{3} + \frac{1}{5}\right)\frac{\tan^6 x}{6^2} - \&c$

$= \frac{x^2}{2}\log_e \tan x + x\left(\frac{\sin 2x}{1^2} + \frac{\sin 6x}{3^2} + \frac{\sin 10x}{5^2} + \&c\right)$

$+ \frac{1}{2}\left(\frac{\cos 2x}{1^3} + \frac{\cos 6x}{3^3} + \frac{\cos 10x}{5^3} + \&c\right) - \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$

Ex. $\frac{1}{1^2} - \frac{1+\frac{1}{3}}{4^2} + \frac{1+\frac{1}{3}+\frac{1}{5}}{6^2} - \&c$

$= \frac{\pi}{4}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c\right) - \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$

24. $\frac{\cos^2 x + \sin^2 x}{2^2} + \frac{2}{3}\cdot\frac{\cos^4 x + \sin^4 x}{4^2} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{\cos^6 x + \sin^6 x}{6^2}$

$+ \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{\cos^8 x + \sin^8 x}{8^2} + \&c$

$= -\frac{\pi^2}{8}\log_e 2\cos x + \frac{\pi}{4}\left\{\frac{\cos x}{1^2} + \frac{1}{2}\cdot\frac{\cos^3 x}{3^2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\cos^5 x}{5^2} + \&c\right\}$

$+ \frac{1}{4}\left\{\frac{\sin^2 2x}{2^2} + \frac{2}{3}\cdot\frac{\sin^4 2x}{4^2} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{\sin^6 2x}{6^2} + \&c\right\}$

$- \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right).$

25. $\frac{\tan^2 x}{2^2} - \left(1 + \frac{1}{3}\right)\frac{\tan^4 x}{4^2} + \left(1 + \frac{1}{3} + \frac{1}{5}\right)\frac{\tan^6 x}{6^2} - \&c$

$= 2\left\{\frac{\sin^2 x}{2^2} + \frac{2}{3}\cdot\frac{\sin^4 x}{4^2} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{\sin^6 x}{6^2} + \&c\right\}$

$- \frac{1}{4}\left\{\frac{\sin^2 2x}{2^2} + \frac{2}{3}\cdot\frac{\sin^4 2x}{4^2} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{\sin^6 2x}{6^2} + \&c\right\}$

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$$\int_0^{\alpha} \frac{d\phi}{\sqrt{(1-a\sin^2\phi)(1-b\sin^2\phi)}} = \frac{1}{\sqrt{1-b}} \int_0^{\beta} \frac{d\phi}{\sqrt{1-\frac{a-b}{1-b}\sin^2\phi}}$$

$$\frac{\tan\beta}{\tan\alpha} = \sqrt{1-b}.$$

(i) $\sqrt[4]{y}\,(1-y)(1-y^2)(1-y^3)$ &c $= \frac{\sqrt[4]{x}\,\sqrt[8]{1-x}}{\sqrt[8]{3}}\sqrt{1+\frac{1\cdot 7}{3^2}x+\&c}$

(ii) $= \frac{\sqrt[4]{x}\,\sqrt[12]{1-x}}{\sqrt[6]{2}}\sqrt{1+\frac{1\cdot 3}{4^2}x+\&c}$ (iii) $= \sqrt[24]{\frac{x(1-x)}{48^2}}\sqrt{1+\frac{1\cdot 5}{6^2}x+\&c}$

$\sqrt[12]{y}\,(1-y^2)(1-y^4)(1-y^6)\,\&c = \frac{\sqrt[12]{x}\,\sqrt[4]{1-x}}{\sqrt{2}}\sqrt{1+\frac{1\cdot 3}{4^2}x+\&c}$

$\sqrt[8]{y}\,(1-y^3)(1-y^6)(1-y^9)\,\&c = \sqrt[8]{\frac{x}{27}}\sqrt[4]{1-x}\sqrt{1+\frac{1\cdot 2}{3^2}x+\&c}$

$1+240\left(\frac{1^3y^3}{1-y^3}+\frac{2^3y^6}{1-y^6}+\&c\right) = \left(1+\frac{1\cdot 2}{3^2}x+\&c\right)^4\left(1-\frac{8}{9}x\right)$

$1-504\left(\frac{1^5y^3}{1-y^3}+\frac{2^5y^6}{1-y^6}+\&c\right) = \left(1+\frac{1\cdot 2}{3^2}x+\&c\right)^6\left(1-\frac{4}{3}x+\frac{8}{27}x^2\right)$

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26. i. $\int_0^{\frac{\pi}{2}} x \cos^n x \sin nx\,dx = \frac{\pi}{2^{n+2}}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}\right)$. 82

ii. $\int_0^{\frac{\pi}{2}} \cos^n x \sin nx\,dx = \frac{1}{2^{n+1}}\left(\frac{2}{1}+\frac{2^2}{2}+\cdots+\frac{2^n}{n}\right)$.

These are true for all values n.

Cor. 1. Shew that $\frac{2-1}{1}+\frac{2^2-1}{2}+\frac{2^3-1}{3}+\cdots+\frac{2^x-1}{x}$ can be expanded in ascending powers of x in a Convergent series, the first two terms being $\frac{S_2}{2}x+\frac{S_3}{8}x^2+\&c$.

2. If $\phi(x)=\frac{2-1}{1}+\frac{2^2-1}{2}+\cdots+\frac{2^x-1}{x}$, then shew that

$\phi(-x)+\Sigma\frac{1}{x-1}+\frac{1}{2^x}\cdot\frac{1}{x}+\frac{1}{2^{x+1}}\cdot\frac{1}{1+x}+\frac{1}{2^{x+2}}\cdot\frac{1}{2+x}+\&c$

hence shew how to find the value of the series

$\frac{1}{1^n}\frac{1}{2}+\frac{1}{2^n}\frac{1}{2^2}+\frac{1}{3^n}\frac{1}{2^3}+\&c$

27. $1^n\log_e 1+2^n\log_e 2+3^n\log_e 3+\cdots+x^n\log_e x=\phi_n(x)$

$\phi_n(x)=C_n+(1^n+2^n+3^n+\cdots+x^n-S_{-n})\log_e x-\frac{x^{n+1}}{(n+1)^2}$

$+\frac{B_2}{\lfloor 2}\cdot n\cdot\frac{1}{n}\cdot x^{n-1}-\frac{B_4}{\lfloor 4}n(n-1)(n-2)\left(\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}\right)x^{n-3}+\&$

and $C_{n-1}=\frac{B_n}{n}\left\{\cos\frac{\pi n}{2}\left(\Sigma\frac{1}{n-1}-C_0-\log_e 2\pi\right)-\frac{\pi}{2}\sin\frac{\pi n}{2}\right.$

$\left.-\frac{2\lfloor n-1}{(2\pi)^n}\cos\frac{\pi n}{2}\left\{\frac{\log_e 1}{1^n}+\frac{\log_e 2}{2^n}+\frac{\log_e 3}{3^n}+\&c\right\}\right.$

Cor. If n is even $C_n=-\frac{\pi}{2}\frac{B_{n+1}}{n+1}\cos\frac{\pi n}{2}=-\frac{\lfloor e}{2(2\pi)^n}S_{n+1}$.

$C_0=\frac{1}{2}\log_e 2\pi$, $C_2=\frac{S_3}{4\pi^2}$, $C_4=-\frac{35_5}{4\pi^4}$, $C_6=\frac{45_7}{8\pi^6}$, &c

Sol. Divide both sides in IX. by x^n and then differentiate both sides with regards to n,

Ex. 1. $\frac{(1^1.2^2.3^3\ldots x^x)^2 e^{\frac{1}{6}(x^2+\frac{1}{x})}}{\sqrt[3]{\frac{x^{x(x+3)}\lfloor x^2}{x+\frac{1}{4}}}}$ when $x=\infty$

$=1\cdot\frac{1}{\pi^2}\cdot 2^{\frac{1}{(2\pi)^2}}\cdot 3^{\frac{1}{(3\pi)^4}}\cdot 4^{\frac{1}{(4\pi)^2}}\cdot 5^{\frac{1}{(5\pi)^4}}\&c$

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$\phi(1) - \phi(2) + \phi(3) - \&c.$

$= \phi(0) - \phi(-1) + \phi(-2) - \&c$

If $\alpha\beta = \frac{\pi}{4}$ then

$\sqrt{\alpha}\left\{1 - \frac{\alpha^2}{12} + \frac{\alpha^4}{160} - \frac{61}{120}\alpha^6 + \&c\right\}$

$= \sqrt{\beta}\left\{1 - \frac{\beta^2}{12} + \frac{\beta^4}{160} - \frac{61}{120}\beta^6 + \&c\right\}$

~~$\sqrt{\alpha}\left\{e^{\alpha^2} - e^{9\alpha^2} + e^{25\alpha^2} - \&c\right\} = \sqrt{\beta}\left\{e^{\beta^2} - e^{9\beta^2} + e^{25\beta^2} - \&c\right\}$~~

$$\int_0^\infty e^{-x^2} \sin 2nx\, dx = \frac{\sqrt{\pi}}{2} e^{-n^2}$$

If $\alpha\beta = \pi$, then

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}}{e^{\alpha x} + e^{-\alpha x}}\, dx = \sqrt{\beta}\int_0^\infty \frac{e^{-x^2}}{e^{\beta x} + e^{-\beta x}}\, dx$$

$$1 - xa + x^3a^2 - x^6a^3 + x^{10}a^4 - \&c$$

$$= \cfrac{1}{1 + \cfrac{xa}{1 + \cfrac{(x^2-x)a}{1 + \cfrac{x^3a}{1 + \cfrac{(x^4-x^2)a}{1 + \cfrac{x^5a}{1 + \&c}}}}}}$$

$$D_{2n\alpha} = 1 + a x^n \cdot \frac{1-x^n}{1-x} + a^2 x^{2n} \cdot \frac{(1-x^n)(1-x^{n-1})}{(1-x)(1-x^2)}$$

$$+ a^3 x^{3n} \cdot \frac{(1-x^n)(1-x^{n-1})(1-x^{n-2})}{(1-x)(1-x^2)(1-x^3)} + \&c = \phi(a)$$

$D_{2n+1} = \phi(ax).$

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2. $\left\{\left(\frac{1}{x}\right)^{1^2}\cdot\left(\frac{2}{x}\right)^{2^2}\cdot\left(\frac{3}{x}\right)^{3^2}\left(\frac{4}{x}\right)^{4^2}\cdots\left(\frac{x}{x}\right)^{x^2}\right\}\cdot e^{\frac{x^3}{9}-\frac{x}{12}}$ when $x=\infty$

$\therefore e^{\frac{2}{\pi^2}\left(\frac{1}{1^3}+\frac{1}{4^3}+\frac{1}{6^3}+\frac{1}{8^3}+\&c\right)}$

28. $\phi_n(x)+n\,C_{n-1}x+\frac{n(n-1)}{\lfloor 2}C_{n-2}x^2+\frac{n(n-1)(n-2)}{\lfloor 3}C_{n-3}x^3+$

$\cdots+C_0x^n+S_1\frac{x^{n+1}}{n+1}\log-S_2\frac{x^{n+2}\lfloor 1}{(n+1)(n+2)}+\&c=f(x,n)$

where c_n is the constant of $1^n\log 1+2^n\log 2+\&c$

& S_n is that of $\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\&c$.

$$f(x,n+1)=\frac{1^n+2^n+3^n+\cdots+x^n}{n}+n\int_0^x f(x,n-1)\,dx$$

$$f(x,n)=(1^n+2^n+\cdots x^n-S_{-n})\Sigma\frac{1}{n}-\frac{n}{\lfloor 1}B_1x^{n-1}+$$

$$\frac{n(n-1)(n-2)}{\lfloor 4}B_4'x^{n-3}\left(1+\frac{1}{2}+\frac{1}{3}\right)-\frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 6}B_6$$

$$x^{n-5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)+\&c$$

29. $\phi_n(x)-n^n\left\{\phi_n\left(\frac{x}{n}\right)+\phi_n\left(\frac{x-1}{n}\right)+\cdots+\phi_n\left(\frac{x-n+1}{n}\right)\right\}$

$=(1^n+2^n+\cdots x^n)\log_e n-S_{-n}\log_e n-(n^{n+1}-1)C_n$.

Cor 1. $\phi_n\left(-\frac{1}{n}\right)+\phi_n\left(-\frac{2}{n}\right)+\phi_n\left(-\frac{3}{n}\right)+\cdots+\phi_n\left(-\frac{n-1}{n}\right)$

$=\frac{\log_e n}{n^n}S_{-n}+\left(n-\frac{1}{n^n}\right)C_n$

Cor 2. $\phi_n\left(-\frac{1}{2}\right)=\frac{\log_e 2}{2^n}S_{-n}+\left(2-\frac{1}{2^n}\right)C_n$.

30. If n is even

$$\phi_n(x-1)+\phi_n(-x)=2C_n+\frac{\lfloor n}{(2\pi)^n}\cos\frac{\pi n}{2}\left\{\frac{\cos 2\pi x}{1^{n+1}}+\frac{\cos 4\pi x}{2^{n+1}}\right.$$

If n is odd

$$\phi_n(x-1)-\phi_n(-x)=\frac{\lfloor n}{(2\pi)^n}\sin\frac{\pi n}{2}\left\{\frac{\sin 2\pi x}{1^{n+1}}+\frac{\sin 4\pi x}{2^{n+1}}\right.$$

Sol. $\Sigma\frac{1}{x-1}-\Sigma\frac{1}{-x}=-\pi\cot-\pi x=-\frac{\pi}{2}(\sin 2\pi x+\sin 4\pi x$

Integrate both sides $x+1$ times.

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If $\int_0^h \phi(x) \cos nx \, dx = \psi(n)$ & $\alpha/\beta = \pi$

then $\alpha \left\{ \frac{1}{2}\phi(0) + \phi(\alpha)\cos n\alpha + \phi(2\alpha)\cos 2n\alpha + \right.$
$\left. + \phi(n\alpha)\cos mn\alpha \right\}$
$= \psi(n) + \psi(2\beta - n) + \psi(2\beta + n) + \psi(4\beta - n) + \psi(4\beta + n)$
$+ \psi(6\beta - n) + \psi(6\beta + n) + \&c$ ad. inf.

where $m\alpha$ is the greatest multiple of α less than h; but if h be a multiple of α the last term is $\frac{1}{2}\phi(h)\cos nh$; in both the cases α is less than β

$\int_0^h \frac{\sin nx}{\sin x} \phi(x) \, dx$
$= \pi \left\{ \frac{1}{2}\phi(0) - \phi(\pi)\cos n\pi + \phi(2\pi)\cos 2n\pi - \right.$
$\left. \cdots \pm \phi(n\pi)\cos mn\pi. \right\}$
$- 2\psi(n+1) - 2\psi(n+3) - 2\psi(n+5) - \&c$ ad. inf.

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N. B. More general theorems true for all values of n. can be got by differentiating IX. 14 & 15 with respect to n.

31. If $1\log 1 + 2\log 2 + \cdots + x\log x = \phi_1(x)$, then

i. $\pi\{\phi_1(x-1) - \phi_1(-x)\} = \psi(x) - \pi x \log_e(2\sin \pi x)$ where

$$\psi(x) = \sin \pi x + \frac{1}{2}\cdot\frac{\sin^3 \pi x}{3^2} + \frac{1.3}{2.4}\cdot\frac{\sin^5 \pi x}{5^2} + \&c$$

$$= \tan \pi x - (1 + \tfrac{1}{3})\frac{\tan^3 \pi x}{3} + (1 + \tfrac{1}{3} + \tfrac{1}{5})\frac{\tan^5 \pi x}{5} - \&c$$

ii. $\psi(x) + \psi(\tfrac{1}{2} - x) = \tan \pi x - \frac{\tan^3 \pi x}{3^2} + \frac{\tan^5 \pi x}{5^2} \ldots \&c$
$+ \frac{\pi}{2}\log_e(2\cos \pi x)$

iii. $\psi(\tfrac{1}{2} - x) + \frac{1}{2}\psi(2x) - \psi(x) = \frac{\pi}{2}\log_e(2\cos \pi x)$.

Ex. 1. $\psi(\frac{1}{2}) = \frac{\pi}{2}\log_e 2$.

2. $\psi(\frac{1}{4}) = \left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c\right) + \frac{\pi}{4}\log_e 2$.

3. $\psi(\frac{1}{6}) = \frac{3\sqrt{3}}{4}\left(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \&c\right) - \frac{\pi^2}{6\sqrt{3}}$

4. $\psi(\frac{1}{3}) = \frac{\sqrt{3}}{2}\left(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \&c\right) - \frac{\pi^2}{9\sqrt{3}} + \frac{\pi}{6}\log_e 3$.

5. $\psi(\frac{1}{2} - x) + \psi(\frac{1}{2} + x) = \pi\log_e(2\cos \pi x)$

6. Find $\psi(\frac{2}{3})$, $\psi(\frac{1}{4})$, $\psi(\frac{5}{12})$, $\psi(\frac{3}{8}) - \psi(\frac{1}{8})$

7. Find $\psi(\frac{1}{8})$ & $\psi(\frac{1}{12})$

8. $2\psi(x) - \frac{1}{2}\psi(2x) = \tan \pi x - \frac{\tan^3 \pi x}{3^2} + \frac{\tan^5 \pi x}{5^2} \ldots \&c$

Similarly we can find peculiarities for $\phi_2(x)$, $\phi_3(x)$ &c

32. $\sin 2x + \frac{2}{3^2}\sin^3 2x + \frac{2.4}{3.5^2}\sin^5 2x + \frac{2.4.6}{3.5.7^2}\sin^7 2x + \&c$

$= 2\left(\frac{\tan x}{1^2} - \frac{\tan^3 x}{3^2} + \frac{\tan^5 x}{5^2} - \&c\right)$

Cor. $\frac{2x}{1+x^2} + \frac{2}{3^2}\cdot\left(\frac{2x}{1+x^2}\right)^3 + \frac{2.4}{3.5^2}\left(\frac{2x}{1+x^2}\right)^5 + \&c$

$= 2\left(\frac{x}{1^2} - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \&c\right)$.

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If $\dfrac{a_n}{b_n x + \dfrac{a_{n+1}}{b_{n+1} x + \dfrac{a_{n+2}}{b_{n+2} x + \&c}}} = C_n \left(1 - P_n x + Q_n x^2 - R_n x^3 + \&c\right)$

then $C_n C_{n+1} = a_n$; $P_n + P_{n+1} = \dfrac{b_n}{C_{n+1}}$ or $\dfrac{b_n C_n}{a_n}$.

$Q_n + Q_{n+1} = (P_n)^2 \cdot R_n + R_{n+1} = P_n (Q_n - Q_{n+1})$.

$S_n + S_{n+1} = P_n (R_n - R_{n+1}) - Q_n Q_{n+1}$.

Generally

$Z_n + Z_{n+1} = P_n (Y_n - Y_{n+1}) - Q_n X_{n+1} - R_n W_{n+1}$

$- S_n V_{n+1} - \ldots\ldots - X_n Q_{n+1}$.

If $\displaystyle\int_0^\alpha \frac{d\theta}{\sqrt{1 - x \sin^2\theta}} + \int_0^\beta \frac{d\theta}{\sqrt{1 - x \sin^2\theta}} = \int_0^\gamma \frac{d\theta}{\sqrt{1 - x \sin^2\theta}}$

then $\tan \dfrac{\gamma}{2} = \dfrac{\sin\alpha \sqrt{1 - x \sin^2\beta} + \sin\beta \sqrt{1 - x \sin^2\alpha}}{\cos\alpha + \cos\beta}$.

or $\cot\alpha \cot\beta = \dfrac{\cos\gamma}{\sin\alpha \sin\beta} + \sqrt{1 - x \sin^2\gamma}$

$\dfrac{\sqrt{x}}{2} = \dfrac{\sqrt{\sin\delta \cdot \sin(\delta - \alpha) \sin(\delta - \beta) \sin(\delta - \gamma)}}{\sin\alpha \sin\beta \sin\gamma}$ where

$2\delta = \alpha + \beta + \gamma$.

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Ex. 1. $1+\frac{2}{3^2}+\frac{2.4}{3.5^2}+\frac{2.4.6}{3.5.7^2}+\&c = 2\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\&c\right)$

2. $1+\frac{2}{5^2}\cdot\frac{3}{4}+\frac{2.4}{3.5^2}\cdot\left(\frac{3}{4}\right)^2+\frac{2.4.6}{3.5.7^2}\cdot\left(\frac{3}{4}\right)^3+\&c$

$= -\frac{\pi}{3\sqrt{3}}\log_e 3 - \frac{10}{27}\pi^2 + 5\left(\frac{1}{1^2}+\frac{1}{4^2}+\frac{1}{7^2}+\&c\right)$

3. $\frac{1}{2}+\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2.4}{3.5^2}\cdot\frac{1}{2^5}+\frac{2.4.6}{3.5.7^2}\cdot\frac{1}{2^7}+\&c$

$= -\frac{\pi}{6}\log_e(2+\sqrt{3}) + \frac{4}{3}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\&c\right)$

4. $1+\frac{2}{3^2}\cdot\frac{1}{2}+\frac{2.4}{3.5^2}\cdot\frac{1}{2^2}+\frac{2.4.6}{3.5.7^2}\cdot\frac{1}{2^3}+\&c$

$= -\frac{\pi}{2\sqrt{2}}\log_e(1+\sqrt{2}) - \frac{\pi^2}{4\sqrt{2}} + 4\left(\frac{1}{1^2}-\frac{1}{5^2}+\frac{1}{9^2}-\frac{1}{13^2}+\&c\right)$

5. $\left(1-\frac{3}{4}\right)+\frac{2}{3^2}\left(1-\frac{3}{4^2}\right)+\frac{2.4}{3.5^2}\left(1-\frac{3}{4^3}\right)+\frac{2.4.6}{3.5.7^2}\left(1-\frac{3}{4^4}\right)+\&c$

$= \frac{\pi}{4}\log_e(2+\sqrt{3})$

6. $\frac{2x}{1-x^2}-\frac{2}{3^2}\cdot\left(\frac{2x}{1-x^2}\right)^3+\frac{2.4}{3.5^2}\cdot\left(\frac{2x}{1-x^2}\right)^5-\&c$

$= 2\left(\frac{x}{1^2}+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\frac{x^7}{7^2}+\&c\right)$

7. $1-\frac{2}{3^2}+\frac{2.4}{3.5^2}-\frac{2.4.6}{3.5.7^2}+\&c = \frac{\pi^2}{8}-\frac{1}{2}\left\{\log_e(1+\sqrt{2})\right\}^2$

8. $\frac{1}{2}-\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2.4}{3.5^2}\cdot\frac{1}{2^5}-\frac{2.4.6}{3.5.7^2}\cdot\frac{1}{2^7}+\&c$

$= \frac{\pi^2}{12}-\frac{3}{2}\left(\log_e\frac{\sqrt{5}+1}{2}\right)^2$

9. $\tan 2x - \frac{2}{3^2}\tan^3 2x + \frac{2.4}{3.6^2}\cdot\tan^5 2x - \&c$

$= 2\left(\tan x + \frac{\tan^3 x}{3^2} + \frac{\tan^5 x}{5^2} + \&c\right)$

10. $\left(\frac{x}{1+x}\right)+\frac{1}{3^2}\cdot\left(\frac{x}{1+x}\right)^2+\frac{1}{5^2}\cdot\left(\frac{x}{1+x}\right)^3+\frac{1}{7^2}\cdot\left(\frac{x}{1+x}\right)^4+\&c$

$= x-\frac{2}{3}\left(1+\frac{1}{3}\right)x^2+\frac{2.4}{3.5}\left(1+\frac{1}{3}+\frac{1}{5}\right)x^3-\&c$

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If $\frac{\tan\alpha}{\tan\beta} = \sqrt{1-x}$ then

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{1-x\cos^2\theta}} = \int_0^{\beta} \frac{d\theta}{\sqrt{1-x\sin^2\theta}}.$$

$$\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1+x\sin\theta}} = \int_0^{\frac{\pi}{2}} \frac{\cos^{-1}(x\sin^2\theta)}{\sqrt{1-x^2\sin^4\theta}}\, d\theta.$$

$$\left\{\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-x\sin^2\theta}}\right\}^2 = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta\, d\phi}{\sqrt{(1-x\sin^2\theta)(1-x\sin^2\phi\sin^2\theta)}}$$

If $z = 1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{1\cdot 2}\right)^2 x^2 + \&c$

& $y = \pi\, \frac{1+\left(\frac{1}{2}\right)^2(1-x) + \&c}{1+\left(\frac{1}{2}\right)^2 x + \&c}$

then $\frac{1}{2} + \frac{\operatorname{Sech} y}{1+n^2} + \frac{\operatorname{Sech} 2y}{1+(2n)^2} + \frac{\operatorname{Sech} 3y}{1+(3n)^2} + \&c$

$$= \frac{z}{2} + \frac{(nz)^2 x}{2} + \frac{(2nz)^2}{2} + \frac{(3nz)^2 x}{2} + \frac{(4nz)^2}{2} + \&c$$

$$\int_0^{\infty} e^{-n}\int_0^{\phi} \frac{d\theta}{\sqrt{1-x\sin^2\theta}}\, d\phi = \frac{1}{n} + \frac{x}{n} + \frac{2}{n} + \frac{9x}{n} + \&c$$

If $\frac{\sin\alpha}{\sin\beta} = \sqrt{x}$, then

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{x-\sin^2\theta}} = \int_0^{\beta} \frac{d\theta}{\sqrt{1-x\sin^2\theta}}$$

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CHAPTER XII

1. If x, y, z, u & n are positive integers, then

$$n\cdot\frac{\lfloor x+n\rfloor\lfloor y+n\rfloor\lfloor z+n\rfloor\lfloor u+n\rfloor\lfloor x+y+z+n\rfloor\lfloor y+z+u+n\rfloor\lfloor z+u+x+n\rfloor\lfloor u+x+y+n\rfloor}{\lfloor n\rfloor\lfloor x+y+n\rfloor\lfloor y+z+n\rfloor\lfloor z+u+n\rfloor\lfloor u+x+n\rfloor\lfloor x+z+n\rfloor\lfloor y+u+n\rfloor\lfloor x+y+z+u+n\rfloor}$$

$$= n - (n+2)\frac{n}{\lfloor 2}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{u}{u+n+1}\cdot\frac{x+y+z+u+2n+1}{x+y+z+u+n}$$

$$+(n+4)\cdot\frac{n(n+1)}{\lfloor 2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$$\times\frac{u(u-1)}{(u+n+1)(u+n+2)}\cdot\frac{(x+y+z+u+2n+1)(x+y+z+u+2n+2)}{(x+y+z+u+n)(x+y+z+u+n+1)}$$

$$-\ \&c$$

N. B. The above result is not true for all values of x, y, z, u and n. For example it is not true when $x+y+z+u+n=0$ owing to the extraneous factor containing all the quantities x, y, z & u in each term. Unless we get rid of this factor identities deduced from the above won't be true for all values. The only way to get rid of this is to make u infinitely great. The solution of this theorem is evident from the result.

2. $\Sigma\frac{1}{x+n}+\Sigma\frac{1}{y+n}+\Sigma\frac{1}{z+n}-\Sigma\frac{1}{x+y+z}-\Sigma\frac{1}{y+z+n}-$

$$\Sigma\frac{1}{z+x+n}+\Sigma\frac{1}{x+y+z+n}-\Sigma\frac{1}{n}$$

$$=\left(1+\frac{1}{n+1}\right)\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{x+y+z+n+1}{x+y+z+n}$$

$$+\left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$$\times\frac{(x+y+z+2n+1)(x+y+z+2n+2)}{(x+y+z+n)(x+y+z+n+1)}+\&c$$

This is true only for positive integral values.

Sol. Subtract both sides in XII 1 from n then divide both sides by n and then put $n=0$.

3. If x, y, z and n are positive integers then

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If $\frac{\sin\beta}{\sin\alpha} = \frac{1+x}{1+x\sin^2\alpha}$,

then $(1+x)\int_0^\alpha \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\theta}}$.

$(1+x)\int_0^\alpha \frac{d\theta}{\sqrt{1-x^3\frac{2+x}{1+2x}\sin^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-x^3\left(\frac{2+x}{1+2x}\right)^3\sin^2\theta}}$.

If $\frac{1+\sin\beta}{1-\sin\beta} = \frac{1+\sin\alpha}{1-\sin\alpha}\cdot\left(\frac{1+x\sin\alpha}{1-x\sin\alpha}\right)^2$

$(1+x)^2\int_0^\alpha \frac{d\theta}{\sqrt{1-x^4\sin^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-\left\{1-\left(\frac{1-x}{1+x}\right)^4\right\}\sin^2\theta}}$

If $\frac{1+\sin\beta}{1-\sin\beta} = \frac{1+\sin\alpha}{1-\sin\alpha}\left(\frac{1+x\sin\alpha}{1-x\sin\alpha}\right)^2\cdot\frac{1+x^2\sin\alpha}{1-x^2\sin\alpha}$.

If $\sin(2\beta-\alpha) = x\sin\alpha$

$(1+x)\int_0^\alpha \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = 2\int_0^\beta \frac{d\theta}{\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\theta}}$

If $\frac{\tan\alpha}{\tan\beta} = \sqrt{1+x}$

$\int_0^\alpha \frac{d\theta}{\sqrt{1+x\cos^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}}$.

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$$\frac{\lfloor n \lfloor x+y+n \lfloor y+z+n \lfloor z+x+n}{\lfloor x+n \lfloor y+n \lfloor z+n \lfloor x+y+z+n} = 1 + \frac{xyz}{\lfloor\!\lfloor (n+1)(x+y+z+n)} + \frac{x(x-1)\ y(y-1)\ z(z-1)}{\lfloor 2\,(n+1)(n+2)(x+y+z+n)(x+y+z+n-1)} + \&c$$

Sol. Divide both sides in XII by n, write $-n+m$ for a and then make n infinitely great.

If x, y, z, and n are positive integers, then

$$\frac{(x+n)(y+n)(z+n)(x+y+z+n)}{(x+y+n)(y+z+n)(z+x+n)} = n + (n+2)\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{x+y+z+2n}{x+y+z+n-1} + (n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)}\ \frac{y(y-1)}{(y+n+1)(y+n+2)}\ \frac{z(z-1)}{(z+n+1)(z+n+2)} \times \frac{(x+y+z+2n)(x+y+z+2n+1)}{(x+y+z+n-1)(x+y+z+n-2)} + \&c$$

Sol. Put $a=-1$ in XII.

Ex. If x is a positive integer shew that

1. $$\frac{(\lfloor x \lfloor 3x-2)^4}{(\lfloor 2x-1)^6 \lfloor 4x-3} = 1 - 3\left(\frac{x-1}{x+1}\right)^4 \frac{4x-1}{4x-3} + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^4 \frac{4x-1}{4x-3}\cdot\frac{4x}{4x-4} - 7\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\cdot\frac{x-3}{x+3}\right)^4 \frac{4x-1}{4x-3}\cdot\frac{4x}{4x-4}\cdot\frac{4x+1}{4x-5} + \&c$$

2. $$\frac{3}{2} \le \frac{1}{\frac{x-1}{2}} - \frac{3}{2} \le \frac{1}{x-1} + \frac{1}{2} \le \frac{1}{\frac{3x-3}{2}}$$

$$= 1\left(\frac{x-1}{x+1}\right)^3 \frac{3x-1}{3x-3} + \frac{1}{2}\left(\frac{x-1}{x+1}\cdot\frac{x-3}{x+3}\right)^3 \frac{3x-1}{3x-3}\cdot\frac{3x+1}{3x-5} + \&c$$

3. $$\left(\frac{x}{2x-1}\right)^3 (3x-2) = 1 + 3\left(\frac{x-1}{x+1}\right)^3 \frac{3x-1}{3x-3} + 5\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^3 \frac{3x-1}{3x-3}\cdot\frac{3x}{3x+1} + \&c.$$

4. $$\left(\frac{\lfloor 2x}{\lfloor x}\right)^3 \frac{1}{\lfloor 3x} = 1 + \left(\frac{x}{\lfloor\!\lfloor 1}\right)^2 \frac{2x}{3x} + \left\{\frac{x(x-1)}{\lfloor 2}\right\}^2 \frac{x(x-1)}{3x(3x-1)} + \&c.$$

5. $$\frac{8}{9}\cdot\left(\frac{\lfloor 3x}{\lfloor 2x}\right)^3 \frac{\lfloor x}{\lfloor 4x} = 1 + \frac{x}{\lfloor\!\lfloor 1}\cdot\frac{x-1}{x+1}\cdot\frac{x}{4x-1} + \frac{x(x-1)}{\lfloor 2}\cdot\frac{(x+1)(x+2)}{(x+1)(x+2)} \times \frac{x(x-1)}{(4x-1)(4x-2)} + \&c.$$

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$$n + (n+2)\frac{\alpha}{n-\alpha+1}\cdot\frac{\beta}{n-\beta+1} + (n+4)\frac{\alpha(\alpha+1)}{(n-\alpha+1)(n-\alpha+2)}\cdot$$

$$\times\frac{\beta(\beta+1)}{(n-\beta+1)(n-\beta+2)} + \text{&c to } k+1 \text{ terms}$$

$$= \frac{1}{n-\alpha-\beta}\left\{(n-\alpha)(n-\beta) - \frac{\lfloor n-\alpha \rfloor \lfloor n-\beta}{\lfloor \alpha-1 \rfloor \lfloor \beta-1}\cdot\frac{\lfloor \alpha+k \rfloor \lfloor \beta+k}{\lfloor n-\alpha+k \rfloor \lfloor n-\beta+k}\right\}$$

If $\alpha+\beta+\gamma+1=K$ then

$$(K+1)\frac{\lfloor K}{\lfloor 0}\cdot\frac{\lfloor \alpha \lfloor \beta \lfloor \gamma}{\lfloor K-\alpha \lfloor K-\beta \lfloor K-\gamma} + (K+3)\frac{\lfloor K+1}{\lfloor 1}\cdot\frac{\lfloor \alpha+1 \lfloor \beta+1 \lfloor \gamma+1}{\lfloor K-\alpha+1 \lfloor K-\beta+1 \lfloor K-\gamma+1}$$

$$+ (K+5)\frac{\lfloor K+2}{\lfloor 2}\cdot\frac{\lfloor \alpha+2 \lfloor \beta+2 \lfloor \gamma+2}{\lfloor K-\alpha+2 \lfloor K-\beta+2 \lfloor K-\gamma+2} + \text{&c to } n \text{ terms}$$

$$- 2\log_e 2n \quad \text{when } n \text{ becomes infinite.}$$

$$= -\Sigma\frac{1}{\alpha} - \Sigma\frac{1}{\beta} - \Sigma\frac{1}{\gamma} + 2C_1.$$

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5. For all values of x, y, z and n,

$$n\cdot\frac{\lfloor x+n}{\lfloor n}\frac{\lfloor y+n}{\lfloor x+y+n}\frac{\lfloor x+n}{\lfloor y+z+n}\frac{\lfloor x+y+z+n}{\lfloor z+x+n} = n-(n+2)\frac{n}{\lfloor 1}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}$$

$$+(n+4)\frac{n(n+1)}{\lfloor 2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdot\frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$-$ &c.

Sol. Make u infinite in XII 1.

6. $\Sigma\frac{1}{x+n}+\Sigma\frac{1}{y+n}-\Sigma\frac{1}{x+y+n}-\Sigma\frac{1}{n}=\left(1+\frac{1}{n+1}\right)\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}$

$$+\left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}+\&c$$

Sol. Subtract both sides in XII 5 from n, divide both sides by z and then put $z=0$.

7. $\frac{(x+n)(y+n)}{x+y+n}=n+(n+2)\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}+$

$$(n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}+\&c$$

Sol. Put $z=-1$ in XII 5.

8. $n\frac{\lfloor x+y}{\lfloor x}\frac{}{\lfloor y}\cdot\frac{\lfloor x+n}{\lfloor n}\frac{\lfloor y+n}{\lfloor x+y+n}=n+(n+2)\frac{n^2}{\lfloor 1^2}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot$

$$+(n+4)\frac{n^2(n+1)^2}{(\lfloor 2)^2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}+\&c$$

Sol. Write $-n$ for z in XII 5.

9. $\frac{\lfloor x+n}{\lfloor n}\frac{\lfloor y+n}{\lfloor x+y+n}\cdot\frac{\lfloor \frac{n}{2}}{\lfloor x+\frac{n}{2}}\frac{\lfloor x+y+\frac{n}{2}}{\lfloor y+\frac{n}{2}}=1+\frac{n}{\lfloor 1}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}$

$$+\frac{n(n+1)}{\lfloor 2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}+\&c$$

Sol. Put $z=-\frac{n}{2}$ in XII 5.

N. B. This result is very important as the isolated factors n, $n+2$, $n+4$ disappear.

10. $n\frac{\lfloor x+n}{\lfloor n}\frac{\lfloor y+n}{\lfloor x+y+n}\cdot\frac{\lfloor \frac{n}{2}}{\lfloor x+\frac{n}{2}}\frac{\lfloor x+y+\frac{n}{2}}{\lfloor y+\frac{n}{2}}=n+(n+2)\frac{n}{\lfloor 1}\cdot\frac{xy}{(x+n+1)(y+n+1)}$

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$$(1+x)\int_0^{\alpha}\frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = 2\int_0^{\beta}\frac{d\theta}{\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\theta}}$$

~~if~~ $\cancel{(1+x)\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\beta} = \sqrt{1-x^2\sin^2\alpha} + x\cos\alpha}$

$$\int_0^{\alpha}\frac{d\theta}{\sqrt{1-x\sin^2\theta}} + \int_0^{\beta}\frac{d\theta}{\sqrt{1-x\sin^2\theta}} = \int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-x\sin^2\theta}}$$

~~if~~ $\cancel{(1-x\sin^2\alpha)(1-x\sin^2\beta) = 1-x}$

if $\cot\alpha.\cot\beta = \sqrt{1-x}$

$$\int_0^{\alpha}\frac{d\theta}{\sqrt{1-x\sin^2\theta}} = 2\int_0^{\beta}\frac{d\theta}{\sqrt{1-x\sin^2\theta}}$$

if $\frac{\tan\frac{\alpha}{2}}{\tan\beta} = \sqrt{1-x\sin^2\beta}$

$$\int_0^{\sin^{-1}\sqrt[8]{\frac{\beta}{\alpha^3}}}\frac{d\theta}{\sqrt{1-\alpha\sin^2\theta}} = \frac{\pi}{6}\left\{1+\left(\frac{1}{2}\right)^2\alpha + \right\}$$

where $\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} = 1$.

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$+(n+4)\frac{n(n+1)}{\lfloor 2}\frac{x(x-1)}{(x+n+1)(x+n+2)}\frac{y(y-1)}{(y+n+1)(y+n+2)}+\&c$

11. $n\frac{\lfloor x+n\rfloor\lfloor y+n}{\lfloor n\rfloor\lfloor x+y+n}=n-(n+2)\frac{n}{\lfloor 1}\cdot\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}+$

$(n+4)\frac{n(n+1)}{\lfloor 2}\frac{x(x-1)}{(x+n+1)(x+n+2)}\frac{y(y-1)}{(y+n+1)(y+n+2)}\cdots\&c$

12. $\left\{\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\frac{1}{(n+3)^2}+\&c\right\}-\left\{\frac{1}{(x+n+1)^2}+\frac{1}{(x+n+2)^2}+\&c\right\}$

$=\left(1+\frac{1}{n+1}\right)\frac{1}{n+1}\cdot\frac{x}{x+n+1}-\left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{12}{(n+1)(n+2)}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}$

$+\left(\frac{1}{3}+\frac{1}{n+3}\right)\frac{12}{(n+1)(n+2)(n+3)}\frac{x(x-1)(x-2)}{(x+n+1)(x+n+2)(x+n+3)}-\&c$

Sol. Divide both sides in XII 6 by y and then put $y=0$.

13. $\frac{(n-1)(x+n)}{x+n-1}=n-(n+2)\frac{11}{n}\cdot\frac{x}{x+n+1}+$

$(n+4)\frac{12}{n(n+1)}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}-\&c$

14. $0=n-(n+2)\frac{n}{11}\cdot\frac{x}{x+n+1}+(n+4)\frac{n(n+1)}{\lfloor 2}\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot$

$-\&c$

15. $\frac{x+n}{2x+n}=1-\frac{x}{x+n+1}+\frac{x(x-1)}{(x+n+1)(x+n+2)}-\&c$

16. $\frac{(n-1)(x+n)}{2x+n-1}=n-(n+2)\frac{x}{x+n+1}+(n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)}-$

17. $x+n=n+(n+2)\frac{x}{x+n+1}+(n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)}+\&c$

18. $\frac{\lfloor x+n\rfloor\lfloor x-n}{(\lfloor x)^2}\cdot\frac{\sin\pi n}{\pi}=n-(n+2)\frac{n^3}{(11)^3}\cdot\frac{x}{x+n+1}$

$+(n+4)\frac{n^3(n+1)^3}{(\lfloor 2)^3}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}-\&c$

19. $\frac{\lfloor x+n}{\lfloor x\rfloor n}\cdot\frac{\lfloor\frac{n}{2}\rfloor\lfloor x-\frac{n}{2}}{\lfloor-\frac{n}{2}\rfloor\lfloor x+\frac{n}{2}}=1-\frac{n^2}{(11)^2}\frac{x}{x+n+1}$

$+\frac{n^2(n+1)^2}{(\lfloor 2)^2}\frac{x(x-1)}{(x+n+1)(x+n+2)}-\&c$

20. $n\cdot\frac{\lfloor x-\frac{n+1}{2}}{\lfloor x\rfloor-\frac{n+1}{2}}\cdot\frac{\lfloor x+n}{\lfloor x+\frac{n}{2}\rfloor n}=n-(n+2)\frac{n^2}{(11)^2}\frac{x}{x+n+1}$

$+(n+4)\frac{n^2(n+1)^2}{(\lfloor 2)^2}\frac{x(x+1)}{(x+n+1)(x+n+2)}-\&c$

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$$\frac{1}{a+\frac{x^2}{a}} - \frac{2a}{\lfloor 1} \cdot \frac{1}{a+1+\frac{x^2}{a+1}} + \frac{2a(2a+1)}{\lfloor 2} \cdot \frac{1}{a+2+\frac{x^2}{a+2}} - \&c$$

$$= \frac{2\,(\lfloor 2a-1)^2 / \lfloor 2a-1}{\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+2}\right)^2\right\} \&c}$$

$$\int_0^\infty \frac{\phi(xi)+\phi(-xi)}{1+x^2}\,dx = \pi\,\phi(1).$$

If $\int_0^\infty \phi(x)\sin 2n\pi x\,dx = \psi(n)$

then $\alpha\{\phi(\alpha)\sin 2n\alpha + \phi(2\alpha)\sin 4n\alpha + \&\}$

$$= \psi(n) - \psi(\beta-n) + \psi(\beta+n) - \psi(2\beta-n) + \&c$$

with $\alpha\beta = \pi$ & n lying between 0 & β.

$(n-1)$ th $= \sqrt[n]{\alpha\beta} + \sqrt[n]{(1-\alpha)(1-\beta)} = 1$

then $(n-1)^2$ th $= \left(\sqrt[n]{\alpha} - \sqrt[n]{\beta}\right)^n + \left(\sqrt[n]{1-\beta} - \sqrt[n]{1-\alpha}\right)^n$

$$= \left\{\sqrt[n]{\alpha(1-\beta)} - \sqrt[n]{\beta(1-\alpha)}\right\}^n.$$

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21. $\frac{n\lfloor x+n}{\lfloor x \lfloor n} = n + (n+2)\frac{n^2}{(\lfloor 1)^2}\cdot\frac{x}{x+n+1} + (n+4)\frac{n^2(n+1)^2}{(\lfloor 2)^2}\frac{x(x-1)}{(x+n+1)(x+n+2)}$
$+ \&c$

22. $\frac{\lfloor x \lfloor x+n \left(\lfloor \frac{n}{2}\right)^2}{n \lfloor n \left(\lfloor x+\frac{n}{2}\right)^2} = \frac{1}{n} + \frac{n}{\lfloor 1}\cdot\frac{x}{x+n+1}\cdot\frac{1}{n+2}$
$+ \frac{n(n+1)}{\lfloor 2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{1}{n+4} - \&c$

23. $\frac{\lfloor 2x \lfloor x+n}{\lfloor x \lfloor x+n} = 1 - \frac{n}{\lfloor 1}\cdot\frac{x}{x+n+1} + \frac{n(n+1)}{\lfloor 2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} - \&c$

24. $\frac{\lfloor x+n \lfloor \frac{n}{2}}{\lfloor n \lfloor x+\frac{n}{2}} = 1 + \frac{n}{\lfloor 1}\cdot\frac{x}{x+n+1} + \frac{n(n+1)}{\lfloor 2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$

25. $\frac{\lfloor x+n \lfloor \frac{n-1}{2}}{\lfloor n-1 \lfloor x+\frac{n-1}{2}} = n + (n+2)\frac{1}{\lfloor 1}\cdot\frac{x}{x+n+1} +$
$(n+4)\frac{n(n+1)}{\lfloor 2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$

26. $\frac{(n-1)^2}{n-2} = n + (n+2)\frac{1^2}{n^2} + (n+4)\frac{1^2\cdot 2^2}{n^2(n+1)^2} + \&c$

27. $\frac{n-1}{n-2} = 1 + \frac{1}{n} + \frac{1.2}{n(n+1)} + \frac{1.2.3}{n(n+1)(n+2)} + \&c$

28. $\frac{(n-1)^2}{n-3} = n + (n+2)\frac{1}{n} + (n+4)\frac{1.2}{n(n+1)} + \&c$

29. $n-1 = n - (n+2)\frac{1}{n} + (n+4)\frac{1.2}{n(n+1)} - \&c$

30. $\frac{(\lfloor n)^2}{\lfloor 2n}\cdot\frac{\sin\pi n \tan\pi n}{\pi^2 n} = n + (n+2)\frac{n^4}{(\lfloor 4)^4} + (n+4)\frac{n^4(n+1)^4}{(\lfloor 2)^4} + \&c$

31. $\frac{6\left(\lfloor \frac{n}{2}\right)^3 \sin\pi n \sin\frac{\pi n}{2}}{\pi^2 n^2 (1+2\cos\pi n)\lfloor \frac{3n}{2}} = 1 + \frac{n^3}{(\lfloor 1)^3} + \frac{n^3(n+1)^3}{(\lfloor 2)^3} + \&c$

32. $\frac{\lfloor \frac{n-1}{2} \lfloor \frac{-3n+1}{2}}{\left(\lfloor \frac{-n+1}{2}\right)^2}\cdot\frac{\sin\pi n}{\pi} = n + (n+2)\frac{n^3}{(\lfloor 1)^3} + (n+4)\frac{n^3(n+1)^3}{(\lfloor 2)^3} + \&c$

33. $\frac{\sin\pi n}{\pi} = n - (n+2)\frac{n^3}{(\lfloor 1)^3} + (n+4)\frac{n^3(n+1)^3}{(\lfloor 2)^3} - \&c$

34. $\frac{\left(\lfloor \frac{n}{2}\right)^4}{(\lfloor n)^2}\cdot\frac{2\tan\pi n}{\pi n^2} = \frac{1}{n} + \frac{1}{n+2}\cdot\frac{n^2}{(\lfloor 1)^2} + \frac{1}{n+4}\cdot\frac{n^2(n+1)^2}{(\lfloor 2)^2} + \&c$

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$$\int_0^\infty \frac{\cos 2nx\, dx}{\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+2}\right)^2\right\}\cdots} = \frac{\sqrt{\pi}}{2}\cdot\frac{\lfloor a-\frac{1}{2}}{\lfloor a-1}\operatorname{sech}^{2a} n.$$

$$\int_0^\infty \frac{dx}{\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\cdots\left\{1+\left(\frac{x}{b}\right)^2\right\}\left\{1+\left(\frac{x}{b+1}\right)^2\right\}\cdots} = \sqrt{\frac{\pi}{2}}\cdot\frac{\lfloor a-\frac{1}{2}}{\lfloor a-1}\cdot\frac{\lfloor b-\frac{1}{2}}{\lfloor b-1}\cdot\frac{\lfloor a+b-1}{\lfloor a+b-\frac{1}{2}}$$

$$\int_0^\infty \frac{dx}{\left(1+\frac{x^2}{a^2}\right)\left(1+\frac{x^2}{b^2}\right)\left(1+\frac{x^2}{c^2}\right)\left(1+\frac{x^2}{d^2}\right)} = \frac{\pi}{6}\cdot\frac{abcd\left\{(a+b+c+d)^3-(a^3+b^3+c^3+d^3)\right\}}{(a+b)(b+c)(c+a)(a+d)(b+d)(c+d)}$$

If $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$x^4 - px^3 + qx^2 - rx + s = 0$, then

$$\int_0^\infty \frac{dx}{\left(1+\frac{x^2}{\alpha^2}\right)\left(1+\frac{x^2}{\beta^2}\right)\left(1+\frac{x^2}{\gamma^2}\right)\left(1+\frac{x^2}{\delta^2}\right)} = \frac{\pi}{2}\cdot\frac{s}{r-\frac{ps}{q-\frac{r}{p}}}$$

If $\int_0^\infty \phi(a,x)\cos nx\, dx = \psi(a,x)$, then

$$\frac{\pi}{2}\int_0^\infty \phi(a,x)\,\phi(b,x)\,dx = \int_0^\infty \psi(a,x)\,\psi(b,x)\,dx$$

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$\frac{\pi}{2x \lfloor x \sin \pi x} = \frac{1}{x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} \cdot \frac{n(n+1)}{\lfloor 2} + \&c$ 91

36. $\Sigma \frac{1}{x+n} - \Sigma \frac{1}{n} = \left(1 + \frac{1}{n+1}\right) \frac{x}{x+n+1} + \left(\frac{1}{2} + \frac{1}{n+2}\right) \frac{x(x-1)}{(x+n+1)(x+n+2)}$
$+ \&c$

37. $\Sigma \frac{1}{x} + \Sigma \frac{1}{n} - \Sigma \frac{1}{x+n} = \left(1 + \frac{1}{n+1}\right) \frac{n}{\lfloor 1} \cdot \frac{x}{x+n+1} -$
$\left(\frac{1}{2} + \frac{1}{n+2}\right) \frac{n(n+1)}{\lfloor 2} \frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$

38. $2n^2 \left\{ \frac{1}{n^3} + \frac{1}{(n+1)^3} + \frac{1}{(n+2)^3} + \&c \right\}$
$= \left(1 + \frac{1}{n}\right) + \left(\frac{1}{2} + \frac{1}{n+1}\right) \left(\frac{1}{n+1}\right)^2 + \left(\frac{1}{3} + \frac{1}{n+2}\right) \left(\frac{1}{n+1} \cdot \frac{2}{n+2}\right)^2 + \&c$

39. $\left\{ \frac{1}{\left(1 + \frac{n}{2}\right)^2} + \frac{1}{\left(2 + \frac{n}{2}\right)^2} + \&c \right\} - \left\{ \frac{1}{(1+n)^2} + \frac{1}{(2+n)^2} + \&c \right\}$
$= \left(1 - \frac{1}{n+1}\right) \frac{1}{n+1} + \left(\frac{1}{2} - \frac{1}{n+2}\right) \frac{1}{(n+1)(n+2)} + \&c$

40. $\Sigma \frac{1}{n} + \Sigma \frac{1}{-n} = \left(1 + \frac{1}{n+1}\right) \frac{n^2}{(\lfloor 1)^2} + \left(\frac{1}{2} + \frac{1}{n+2}\right) \frac{n^2(n+1)^2}{(\lfloor 2)^2} + \&c$

Ex. 1. $\frac{(\lfloor x)^3 \lfloor 3x-2}{(\lfloor 2x-1)^3} = 1 - 3\left(\frac{x-1}{x+1}\right)^3 + 5 \cdot \left(\frac{x-1}{x+1} \cdot \frac{x-2}{x+2}\right)^3 - \&c$

2. $\frac{x^2}{2x-1} = 1 + 3\left(\frac{x-1}{x+1}\right)^2 + 5\left(\frac{x-1}{x+1} \cdot \frac{x-2}{x+2}\right)^2 + \&c$

3. $\frac{(\lfloor x)^4 \lfloor 4x}{(\lfloor 2x)^4} \cdot \frac{2x}{4x-1} = 1 + \left(\frac{x-1}{x+1}\right)^2 + \left(\frac{x-1}{x+1} \cdot \frac{x-2}{x+2}\right)^2 + \&c$

4. $\frac{(\lfloor x)^2}{\lfloor 2x-1} = 1 - 3\left(\frac{x-1}{x+1}\right)^2 + 5 \cdot \left(\frac{x-1}{x+1} \cdot \frac{x-2}{x+2}\right)^2 - \&c$

5. $x = 1 + 3 \cdot \frac{x-1}{x+1} + 5 \cdot \frac{x-1}{x+1} \cdot \frac{x-2}{x+2} + \&c$

6. $\frac{2^{2x-1}(\lfloor x)^2}{\lfloor 2x} = 1 + \frac{x-1}{x+1} + \frac{x-1}{x+1} \cdot \frac{x-2}{x+2} + \&c$

7. $\frac{x}{2x-1} = 1 - \frac{x-1}{x+1} + \frac{x-1}{x+1} \cdot \frac{x-2}{x+2} - \&c$

8. $0 = 1 - 3\frac{x-1}{x+1} + 5 \cdot \frac{x-1}{x+1} \cdot \frac{x-2}{x+2} - \&c$

9. $\frac{1}{2} \Sigma \frac{1}{x-1} + \frac{(2^{x-1} \lfloor x-1)^2}{\lfloor 2x-1} = 1 + \frac{1}{2} \cdot \frac{x-1}{x+1} + \frac{1}{3} \cdot \frac{x-1}{x+1} \cdot \frac{x-2}{x+2}$

10. $\Sigma \frac{1}{2x} - \Sigma \frac{1}{x} + \frac{1}{2x} = 1 - \frac{1}{2} \cdot \frac{x-1}{x+1} + \frac{1}{3} \cdot \frac{x-1}{x+1}$

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$$\int_0^\infty x^{n-1}\left\{\phi(0) - \frac{x}{\lfloor 1}\phi(1) + \frac{x^2}{\lfloor 2}\phi(2) - \frac{x^3}{\lfloor 3}\phi(3) + \&c\right\} dx$$

$$= \lfloor n-1 \; \phi(-n).$$

$$\lfloor n \int_0^\infty \frac{\lfloor x+p-1}{x^m \lfloor x+p+n} \, dx$$

$$= \frac{\pi}{\sin \pi m}\left\{\frac{1}{\lfloor p^m} - \frac{n}{\lfloor 1}\cdot\frac{1}{(p+1)^m} + \frac{n(n-1)}{\lfloor 2}\cdot\frac{1}{(p+2)^m} - \&c\right\}$$

$$\int_0^\infty \frac{1}{x^{n+1}}\cdot\frac{1-px}{1+x}\cdot\frac{1-pnx}{1+nx}\cdot\frac{1-pn^2x}{1+n^2x}\cdot\frac{1-pn^3x}{1+n^3x} \&c \; dx$$

$$= \cancel{\lfloor n+1} \cdot \frac{1+p}{1+pn^n}\cdot\frac{1+pn}{1+pn^{n+1}}\cdot\frac{1+pn^2}{1+pn^{n+2}}\cdot \&c$$

$$\times \frac{1-n^{n+1}}{1-n}\cdot\frac{1-n^{n+2}}{1-n^2}\cdot\frac{1-n^{n+3}}{1-n^3}\cdot \&c \times \frac{\pi}{\sin \pi n}$$

$$\int_0^\infty \frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+2}\right)^2}\cdot \&c \; dx$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\lfloor a-\frac{1}{2} \; \lfloor b}{\lfloor a-1 \; \lfloor b-\frac{1}{2}}\cdot\frac{\lfloor b-a-\frac{1}{2}}{\lfloor \frac{1}{2}-a}$$

Subtract.

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11. $\frac{2^{4x}(\lfloor x)^2}{4x(\lfloor 2x)^2} = 1 - \frac{2}{3}\cdot\frac{x-1}{x+1} + \frac{2}{5}\cdot\frac{x-2}{x+1}\cdot\frac{x-1}{x+2}$ &c

12. $\frac{1}{2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{x^2}\right) + \frac{1}{2x}\left(\frac{1}{x+1}+\frac{1}{x+2}+\cdots\frac{1}{x+x}\right)$
$= 1 - \frac{1}{2^2}\cdot\frac{x-1}{x+1} + \frac{1}{3^2}\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2} - \frac{1}{4^2}\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\cdot\frac{x-3}{x+3}$ + &c

13. $x(4x-3) = 1^3 + 3^3\frac{x-1}{x+1} + 5^3\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}$ + &c

14. $\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\cdot\left(\frac{1.3}{2.4}\right)^3 - 13\left(\frac{1.3.5}{2.4.6}\right)^3$ + &c

15. $\frac{\pi^2}{8}\left\{1 + 9\left(\frac{1}{4}\right)^4 + 17\left(\frac{1.5}{4.8}\right)^4 + 25\left(\frac{1.5.9}{4.8.12}\right)^4 + \&c\right\}^2$
$= 1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1.3}{2.4}\right)^3 + \left(\frac{1.3.5}{2.4.6}\right)^3$ + &c

16. $1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1.3}{2.4}\right)^3 + \left(\frac{1.3.5}{2.4.6}\right)^3 + \&c = 2\left\{1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 - \&c\right\}^2$

17. $1 + \frac{1}{5}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{9}\cdot\left(\frac{1.3}{2.4}\right)^2 + \frac{1}{13}\left(\frac{1.3.5}{2.4.6}\right)^2$ + &c
$= \frac{\pi}{2}\left\{1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 - \left(\frac{1.3.5}{2.4.6}\right)^2 + \&c\right\}$

18. $\frac{1}{1^2} + \frac{1}{3}\cdot\frac{1}{5^2} + \left(\frac{1.3}{2.4}\right)^2\cdot\frac{1}{9^2} + \&c = \frac{\pi^3}{32}\left\{1 + 9\left(\frac{1}{4}\right)^4 + 17\left(\frac{1.5}{4.8}\right)^4 + \&c\right\}$

41. $\frac{\lfloor n \lfloor x+y+n}{\lfloor x+n \lfloor y+n} = 1 + \frac{x}{11}\cdot\frac{y}{n+1} + \frac{x(x-1)}{12}\cdot\frac{y(y-1)}{(n+1)(n+2)}$ + &c

Sol. Write $-n+m$ for x in XII 5 and make n infinite in equals: the coeff^{ts} of A^x in $(1+A)^{y+n}\left(1+\frac{1}{A}\right)^x = \frac{(1+A)^{x+y+n}}{A^x}$.

42. $\Sigma\frac{1}{a+1} - \Sigma\frac{1}{a-b-1} = \frac{b}{a} + \frac{b(b+1)}{a(a+1)}\cdot\frac{b}{2} + \frac{b(b+1)(b+2)}{a(a+1)(a+2)}\cdot\frac{b}{3}$ + &c

Sol. Subtract 1 from both Sides in XII 41, divide by y and put $y=0$

43. $\frac{\lfloor n \lfloor x}{n\lfloor x+n} = \frac{1}{n} - \frac{x}{11}\cdot\frac{1}{n+1} + \frac{x(x-1)}{12}\cdot\frac{1}{n+2}$ - &c

Sol. Write $-n$ for y in XII 41.

Ex 1. $\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} + \cdots$ &c
$= \frac{1}{n+1} + \frac{11}{2(n+1)(n+2)} + \frac{12}{3(n+1)^2}$

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$$\int_0^\infty \frac{\lfloor x+\alpha}{\lfloor x+\beta}\left(\varepsilon\frac{1}{x+\beta}-\varepsilon\frac{1}{x+\alpha}\right)dx$$

$$= \frac{\lfloor \alpha}{\lfloor \beta}$$ Integration very Simple

$$\int_0^{\frac{\pi}{2}} \cos^m x \cos nx\, dx = \frac{\pi \lfloor m}{2^{m+1}\lfloor\frac{m+n}{2}\lfloor\frac{m-n}{2}}$$

$$\int_0^\infty \frac{\sin^n x}{x^p}\,dx = \frac{1}{\lfloor p-1}\int_0^\infty\int_0^\infty z^{p-1}e^{-zx}\sin^n x\,dz\,dx$$

$$\int_0^\infty e^{-ax}\sin^{2n+1}x\,dx = \frac{\lfloor 2n+1}{(a^2+1^2)(a^2+3^2)\cdots(a^2+(2n+1)^2)}$$

$$\int_0^\infty e^{-ax}\sin^{2n}x\,dx = \frac{\lfloor 2n}{a(a^2+2^2)(a^2+4^2)\cdots(a^2+(2n)^2)}$$

change x to xi then we get $e^x - e^{-x}$

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2. $\frac{\pi}{\sin \pi n} = \frac{1}{n} + \frac{n}{\lfloor 1} \cdot \frac{1}{n+1} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{1}{n+2} + \&c.$

3. $\frac{(2^x \lfloor x)^2}{x \lfloor 2x} = \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{x+1} + \frac{1.3}{2.4} \cdot \frac{1}{x+2} + \frac{1.3.5}{2.4.6} \cdot \frac{1}{x+3} + \&c$

4. $\frac{\pi}{\tan \pi x} \cdot \frac{\lfloor 2x}{(2^x \lfloor x)^2 (1-2x)} = 1 + \frac{x}{1} \cdot \frac{1}{3} + \frac{x(x+1)}{\lfloor 2} \cdot \frac{1}{5} + \&c$

5. $\frac{\lfloor x \lfloor n-1}{\lfloor x+n} \left(\Sigma \frac{1}{x+n} - \Sigma \frac{1}{n-1} \right) = \frac{1}{n^2} - \frac{x}{1} \cdot \frac{1}{(n+1)^2} + \frac{x(x-1)}{\lfloor 2} \cdot \frac{1}{(n+2)^2} - \&c$

6. $\frac{(2^x \lfloor x)^2}{x \lfloor 2x} \left(2\Sigma \frac{1}{2x} - 2\Sigma \frac{1}{x} + \frac{1}{x} - 2 \log 2 \right)$
$= \frac{1}{x^2} + \frac{1}{2} \cdot \frac{1}{(x+1)^2} + \frac{1.3}{2.4} \cdot \frac{1}{(x+2)^2} + \&c$

7. $\frac{\pi}{\tan \pi x} \cdot \frac{\lfloor 2x}{(1-2x)(2^x \lfloor x)^2} \left(\Sigma \frac{1}{2x} - \frac{1}{2} \Sigma \frac{1}{x} + \frac{1}{1-2x} - \frac{\pi}{2} \tan \pi x \right)$
$= \frac{1}{1^2} + \frac{x}{1} \cdot \frac{1}{3^2} + \frac{x(x+1)}{\lfloor 2} \cdot \frac{1}{5^2} + \&c$

8. $-\frac{\pi}{\sin \pi x} \Sigma \frac{1}{n-1} = \frac{1}{n^2} + \frac{n}{1} \cdot \frac{1}{(n+1)^2} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{1}{(n+2)^2} + \&c$

44. $a^n = \{a^n - (b+1)^n\} + \{(a+1)^n - (b+2)^n\} \left(\frac{b+1}{a+1}\right)^n +$
$\{(a+1)^n - (b+3)^n\} \left(\frac{b+1}{a+1}\right)^n \left(\frac{b+2}{a+2}\right)^n + \&c.$

Sol. $a^n = \{a^n - (b+1)^n\} + (b+1)^n = \{a^n - (b+1)^n\} + (a+1)^n \cdot \left(\frac{b+1}{a+1}\right)^n$
$= \&c \ \&c$

Cor 1. $\frac{b}{a-b-1} = \frac{b}{a} + \frac{b(b+1)}{a(a+1)} + \frac{b(b+1)(b+2)}{a(a+1)(a+2)} + \&c$

2. $\frac{b^2}{a-b-1} = (a+b+1) \frac{b^2}{a^2} + (a+b+3) \frac{b^2(b+1)^2}{a^2(a+1)^2} + \&c$

45. If $e^{Ax + A_2 \frac{x^2}{2} + A_3 \frac{x^3}{3} + \&c} = P_0 + P_1 x + P_2 x^2 + \&c$, then
$P_n = P_{n-1} A_1 + P_{n-2} A_2 + P_{n-3} A_3 + \&c$ to n terms and $P_0 = 1$

Sol. Take logarithms of both sides and then differentiate
thrice.

Cor. If $S_n = a_1^n + a_2^n + \dots + a_n^n$ and P_n denotes the nth Sum

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$\frac{ab}{cd} = K$ then

$\frac{1}{2}\{f(a,b)f(c,d) + f(-a,-b)f(-c,-d)\}$

$= f(ac, bd) + ad f(bcK, \frac{bd}{K}) + bc f(bdK, \frac{ac}{K})$

$+ (ad)^3 bc f(acK^2, \frac{bd}{K^2}) + (bc)^3 ad f(bdK^2, \frac{ac}{K^2})$

$+ (ad)^6 (bc)^3 f(bcK^3, \frac{bd}{K^3}) + (bc)^6 (ad)^3 f(bdK^3, \frac{ac}{K^3})$

$+ \&c$

$\frac{1}{2}\{f(a,b)f(c,d) - f(-a,-b)f(-c,-d)\}$

$= a f(\frac{c}{a}, \frac{a}{c}.abcd) + d f(\frac{b}{a}, \frac{d}{b}.abcd)$

$+ a^3 bc f(\frac{c}{aK}, \frac{aK}{c}.abcd) + d^3 bc f(\frac{bK}{d}, \frac{d}{bK}.abcd)$

$+ a^6 d (bc)^3 f(\frac{c}{aK^2}, \frac{aK^2}{c}.abcd) + ad^6 (bc)^3 f(\frac{bK^2}{d}, \frac{d}{bK^2}.abcd)$

$+ \&c \quad \&c \quad \&c$

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of the products of $a_1, a_2, a_3 \cdots a_n$ taken r at a time 94.
then $rP_r = P_{r-1}S_1 - P_{r-2}S_2 + P_{r-3}S_3 - P_{r-4}S_4 + \&c$ and $P_0 = 1$

46. $\frac{1}{n^{2r+1}} - \frac{x}{\lfloor 1} \cdot \frac{1}{(n+1)^{2r+1}} + \frac{x(x-1)}{\lfloor 2} \cdot \frac{1}{(n+2)^{2r+1}} - \&c$

$= \frac{\lfloor x \lfloor n}{n \lfloor x+n} \phi(r)$ where $\phi(0) = 1$ and

$r\phi(r) = S_1\phi(r-1) + S_2\phi(r-2) + S_3\phi(r-3) + \&c$ to r terms

where $S_n = \frac{1}{n^{2r}} - \frac{1}{(x+n+1)^{2r}} + \frac{1}{(n+1)^{2r}} - \frac{1}{(x+n+2)^{2r}} + \&c$

Cor 1. $1 + \frac{1}{2} \cdot \frac{1}{3^{2r+1}} + \frac{1.3}{2.4} \cdot \frac{1}{5^{2r+1}} + \frac{1.3.5}{2.4.6} \cdot \frac{1}{7^{2r+1}} + \&c = \frac{\pi}{2}\phi(r)$

where $\phi(0) = 1$ and $r\phi(r) = S_1\phi(r-1) + S_2\phi(r-2) + S_3\phi(r-3) + \&c$
to r terms where $S_r = \frac{1}{1^{2r}} - \frac{1}{2^{2r}} + \frac{1}{3^{2r}} - \frac{1}{4^{2r}} + \&c$

Cor 2. $\frac{1}{2^{2r+1}} + \frac{1}{2} \cdot \frac{1}{4^{2r+1}} + \frac{1.3}{2.4} \cdot \frac{1}{6^{2r+1}} + \&c = \phi(r)$ where

$\phi(0) = 1$ and $r\phi(r) = S_1\phi(r-1) + S_2\phi(r-2) + \&c$ where

$S_r = \frac{1}{2^{2r}} - \frac{1}{3^{2r}} + \frac{1}{4^{2r}} - \frac{1}{5^{2r}} + \&c.$

Sol. Write $x-P$ for x in XII 43 and equate the coeffts of P.

Ex. $1 + \frac{1}{2} \cdot \frac{1}{3^3} + \frac{1.3}{2.4} \cdot \frac{1}{5^3} + \&c = \frac{\pi^3}{48} + \frac{\pi}{4}(\log 2)^2$

47. $\frac{1}{(x+1)^n} + \frac{1+\frac{1}{2}}{(x+2)^n} + \frac{1+\frac{1}{2}+\frac{1}{3}}{(x+3)^n} + \frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{(x+4)^n} + \&c = \frac{\pi}{2} S_{n+1}$

$-(S_1 S_n + S_2 S_{n-1} + S_3 S_{n-2} + \&c$ the last term being $S_{\frac{n}{2}} S_{n+\frac{1}{2}}$
or $\frac{1}{2} S_{n+\frac{1}{2}} S_{n-\frac{1}{2}}$ according as n is even or odd) where

$S_n = \frac{1}{x^n} + \frac{1}{(x+1)^n} + \frac{1}{(x+2)^n} + \&c$ and $S_1 = \cot^{-1} x$.

Sol. $1(\frac{1}{2} - \frac{1}{n+2}) + (1+\frac{1}{2})(\frac{1}{3} - \frac{1}{n+3}) + (1+\frac{1}{2}+\frac{1}{3})(\frac{1}{4} - \frac{1}{n+4}) + \&c$

$= \frac{1}{2}\left\{(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n})^2 + (\frac{1}{1^2} + \frac{1}{2^2} + \cdots \frac{1}{n^2})\right\}$

In this identity write $n+x-1$ for n and equate
the coeff^{ts} of x^2.

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If $P = \cfrac{l}{x + \cfrac{1^2-n^2}{x + \cfrac{2^2-l^2}{x + \cfrac{3^2-n^2}{x + \cfrac{4^2-l^2}{x+\&c}}}}}$

then $\dfrac{1-P}{1+P} = \dfrac{\left|\frac{x+l+n-3}{4}\right| \left|\frac{x+l-n-3}{4}\right| \left|\frac{x-l+n-1}{4}\right| \left|\frac{x-l-n-1}{4}\right|}{\left|\frac{x-l+n-3}{4}\right| \left|\frac{x-l-n-3}{4}\right| \left|\frac{x+l+n-1}{4}\right| \left|\frac{x+l-n-1}{4}\right|}$

$\left|\frac{x+l+n-3}{4}\right| \left|\frac{x-l+n-3}{4}\right| \left|\frac{x+l-n-3}{4}\right| \left|\frac{x-l-n-3}{4}\right|$

$\left|\frac{x+l+n-1}{4}\right| \left|\frac{x-l+n-1}{4}\right| \left|\frac{x+l-n-1}{4}\right| \left|\frac{x-l-n-1}{4}\right|$

$= \cfrac{8}{\frac{x^2-l^2+n^2-1}{2} + \cfrac{1^2-n^2}{1 + \cfrac{1^2-l^2}{x^2-1} + \cfrac{3^2-n^2}{1 + \cfrac{3^2-l^2}{x^2-1+\&c}}}}$

$\left\{\frac{1}{(x-n+1)^2} + \frac{1}{(x-n+3)^2} + \frac{1}{(x-n+5)^2} + \&c\right\}$

$- \left\{\frac{1}{(x+n+1)^2} + \frac{1}{(x+n+3)^2} + \frac{1}{(x+n+5)^2} + \&c\right\}$

$= \cfrac{n}{x^2-1+n^2} + \cfrac{2(1^2-n^2)}{1 + \cfrac{2}{3(x^2-1)+n^2} + \cfrac{4(2^2-n^4)}{1+\&c}}$

$= \cfrac{n}{x^2-n^2+1 - \cfrac{4(1^2-n^4)}{3(x^2-n^2+5) - \&c}}$

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CHAPTER XIII

1. If $A_0 - nA_1 + \frac{n(n-1)}{\lfloor 2} A_2 - \frac{n(n-1)(n-2)}{\lfloor 3} A_3 + \&c = P_n$, then

$$P_0 + nP_1 + \frac{n(n-1)}{\lfloor 2} P_2 + \frac{n(n-1)(n-2)}{\lfloor 3} P_3 + \&c = A_n.$$

2. $$\frac{A_0}{x^n} + \frac{n}{\lfloor 1} \cdot \frac{A_1}{x^{n+1}} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{A_2}{x^{n+2}} + \&c$$

$$= \frac{A_0}{(x+h)^n} + \frac{n}{\lfloor 1} \cdot \frac{A_1 + hA_0}{(x+h)^{n+1}} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{A_2 + 2hA_1 + h^2 A_0}{(x+h)^{n+2}} + \&c$$

3. If $$\frac{A_0}{x^n} + \frac{n}{\lfloor 1} \cdot \frac{A_1}{x^{n+1}} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{A_2}{x^{n+2}} + \&c$$

$$= \frac{A_0}{(x-1)^n} - \frac{n}{\lfloor 1} \cdot \frac{A_1}{(x-1)^{n+1}} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{A_2}{(x-1)^{n+2}} - \&c,$$ then

(a) $$e^x = \frac{A_0 + \frac{x}{\lfloor 1} A_1 + \frac{x^2}{\lfloor 2} A_2 + \frac{x^3}{\lfloor 3} A_3 + \&c}{A_0 - \frac{x}{\lfloor 1} A_1 + \frac{x^2}{\lfloor 2} A_2 - \frac{x^3}{\lfloor 3} A_3 + \&c}$$

Sol. Multiply both sides in XIII 3 by x^n and make n and x - finite such that $\frac{n}{x} = y$

(b). $$\frac{1}{\{\phi(x)\}^n}\left[A_0 + A_1 \frac{n}{\lfloor 1}\left\{\frac{\phi(x) - \phi(-x)}{\phi(x)}\right\} + A_2 \frac{n(n+1)}{\lfloor 2}\left\{\frac{\phi(x) - \phi(-x)}{\phi(x)}\right\}^2 + \&c\right]$$

is always an even function of x whatever be $\phi(x)$.

Sol. Writing in XIII 3, $\frac{\phi(x)}{\phi(x) - \phi(-x)}$ for x, we have

$$\frac{A_0}{\{\phi(x)\}^n} + A_1 \frac{n}{\lfloor 1} \cdot \frac{\phi(x) - \phi(-x)}{\{\phi(x)\}^{n+1}} + A_2 \frac{n(n+1)}{\lfloor 2} \cdot \frac{\{\phi(x) - \phi(-x)\}^2}{\{\phi(x)\}^{n+2}} + \&c$$

$$= \frac{A_0}{\{\phi(-x)\}^n} + A_1 \frac{n}{\lfloor 1} \cdot \frac{\phi(-x) - \phi(x)}{\{\phi(-x)\}^{n+1}} + A_2 \frac{n(n+1)}{\lfloor 2} \cdot \frac{\{\phi(-x) - \phi(x)\}^2}{\{\phi(-x)\}^{n+2}} + \&c$$

$\therefore$ Each of these is an even function of x.

(c) If n is even, the value of A_{n+1} depends upon the value of A_n. But we may give for A_n any value we choose.

$$\frac{A_{n-1}}{2} = \frac{n-1}{\lfloor 2}(2^2 - 1) B_2 A_{n-2} - \frac{(n-1)(n-2)(n-3)}{\lfloor 4}(2^4 - 1) B_4 A_{n-4} +$$

$$\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{\lfloor 6}(2^6 - 1) B_6 A_{n-6} - \&c.$$

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$$\phi(0) + \frac{m}{n}\cdot\frac{\phi'(0)}{\lfloor 1} + \frac{m(m+1)}{n(n+1)}\frac{\phi''(0)}{\lfloor 2} + \&c.$$

$$= \phi(1) + \frac{m-n}{n}\frac{\phi'(1)}{\lfloor 1} + \frac{(m-n)(m-n-1)}{n(n+1)}\cdot\frac{\phi''(1)}{\lfloor 2} + \&c.$$

$$V = \frac{2lmn}{y+p-2ml^2 + \dfrac{2(1-m)(1^2-n^2)}{1 + \dfrac{2(1+m)(1^2-l^2)}{3y+p+}}}$$

$$\frac{2(2-m)(2^2-n^2)}{1 + \dfrac{2(2+m)(2^2-l^2)}{5y+p+}} \;\&c\;\&c$$

where $y = x^2-(1-m)^2$ & $p = (x^2-l^2)(1-2m)$.

If $\phi(x,y) = x + \dfrac{(1+y)^2+n}{2x + \dfrac{(3+y)^2+n}{2x + \dfrac{(5+y)^2+n}{2x+\&c}}}$

then $\phi(x,y) = \phi(y,x)$.

&c.

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Sol. We have from XIII 3 (a),

$$A_0 + \frac{x}{\lfloor 1} A_1 + \frac{x^2}{\lfloor 2} A_2 + \&c = e^x\left(A_0 - \frac{x}{\lfloor 1} A_1 + \frac{x^2}{\lfloor 2} A_2 - \&c\right)$$

i.e $(e^{-x}+1)\left(A_0 + \frac{x}{\lfloor 1} A_1 + \frac{x^2}{\lfloor 2} A_2 + \&c\right) = (e^x+1)\left(A_0 - \frac{x}{\lfloor 1} A_1 + \frac{x^2}{\lfloor 2} - \&c\right)$

$\therefore \frac{1}{e^x+1}\left(A_0 + \frac{x}{\lfloor 1} A_1 + \frac{x^2}{\lfloor 2} A_2 + \&c\right)$ is an even function of x

Since n is even the coeff^t. of x^{n-1} must be 0.

4. $$\frac{1}{x^r} + \frac{r}{\lfloor 1}\cdot\frac{m}{n}\cdot\frac{1}{x^{r+1}} + \frac{r(r+1)}{\lfloor 2}\cdot\frac{m(m+1)}{n(n+1)}\cdot\frac{1}{x^{r+2}} + \&c$$

$$= \frac{1}{(x-1)^r} + \frac{r}{\lfloor 1}\cdot\frac{m-n}{n}\cdot\frac{1}{(x-1)^{r+1}} + \frac{r(r+1)}{\lfloor 2}\cdot\frac{(m-n)(m-n-1)}{n(n+1)}\cdot\frac{1}{(x-1)^{r+2}} + \&c$$

Sol. We have by

$$\frac{\lfloor n-1 \; \lfloor m+k-1}{\lfloor n+k-1 \; \lfloor m-1} = 1 + \frac{k}{\lfloor 1}\cdot\frac{m-n}{n} + \frac{k(k-1)}{\lfloor 2}\cdot\frac{(m-n)(m-n-1)}{n(n+1)} + \&c$$

multiplying both sides by $\frac{\lfloor r+k-1}{\lfloor r-1 \; \lfloor k}$ we have

$$\frac{\lfloor r+k-1}{\lfloor k \; \lfloor r-1}\cdot\frac{\lfloor n-1 \; \lfloor m+k-1}{\lfloor n+k-1 \; \lfloor m-1} = \frac{\lfloor r+k-1}{\lfloor r-1 \; \lfloor k} + \frac{r}{\lfloor 1}\cdot\frac{m-n}{n}\cdot\frac{\lfloor r+k-1}{\lfloor r \; \lfloor k-1} + \&c$$

$\therefore$ The series in which L.H.S is the coeff^t. of $\frac{1}{x^{r+k}}$

= that in which R.H.S is the coeff^t. of $\frac{1}{x^{r+k}}$

5. $$e^x = \frac{1 + \frac{m}{n}\cdot\frac{x}{\lfloor 1} + \frac{m(m+1)}{n(n+1)}\cdot\frac{x^2}{\lfloor 2} + \frac{m(m+1)(m+2)}{n(n+1)(n+2)}\cdot\frac{x^3}{\lfloor 3} + \&c}{1 + \frac{m-n}{n}\cdot\frac{x}{\lfloor 1} + \frac{(m-n)(m-n-1)}{n(n+1)}\cdot\frac{x^2}{\lfloor 2} + \frac{(m-n)(m-n-1)(m-n-2)}{n(n+1)(n+2)}\cdot\frac{x^3}{\lfloor 3} + \&c}$$

Sol. multiply both sides in XIII 4 by x^r; make x & r infinite such that $\frac{r}{x} = y$.

6. $$\frac{1}{x^r} + \frac{r}{\lfloor 1}\cdot\frac{m}{2m}\cdot\frac{1}{x^{r+1}} + \frac{r(r+1)}{\lfloor 2}\cdot\frac{m(m+1)}{2m(2m+1)}\cdot\frac{1}{x^{r+2}} + \&c$$

$$= \frac{1}{(x-1)^r} - \frac{r}{\lfloor 1}\cdot\frac{m}{2m}\cdot\frac{1}{(x-1)^{r+1}} + \frac{r(r+1)}{\lfloor 2}\cdot\frac{m(m+1)}{2m(2m+1)}\cdot\frac{1}{(x-1)^{r+2}} - \&c$$

7. $$e^x = \frac{1 + \frac{m}{2m}\cdot\frac{x}{\lfloor 1} + \frac{m(m+1)}{2m(2m+1)}\cdot\frac{x^2}{\lfloor 2} + \frac{m(m+1)(m+2)}{2m(2m+1)(2m+2)}\cdot\frac{x^3}{\lfloor 3} + \&c}{1 - \frac{m}{2m}\cdot\frac{x}{\lfloor 1} + \frac{m(m+1)}{2m(2m+1)}\cdot\frac{x^2}{\lfloor 2} - \frac{m(m+1)(m+2)}{2m(2m+1)(2m+2)}\cdot\frac{x^3}{\lfloor 3} + \&c}$$

Ex. 1. $$e^x = \frac{1 + \frac{1}{2}\cdot\frac{x}{\lfloor 1} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^2}{\lfloor 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^3}{\lfloor 3} + \&c}{1 - \frac{1}{2}\cdot\frac{x}{\lfloor 1} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^2}{\lfloor 2} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^3}{\lfloor 3} + \&c}$$

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If $K = \dfrac{\left\lfloor\frac{x+l+m+n-1}{2}\right\rfloor \left\lfloor\frac{x+l-m-n-1}{2}\right\rfloor \left\lfloor\frac{x+m-n-l-1}{2}\right\rfloor \left\lfloor\frac{x+n-l-m}{2}\right\rfloor}{\left\lfloor\frac{x-l-m-n-1}{2}\right\rfloor \left\lfloor\frac{x-l+m+n-1}{2}\right\rfloor \left\lfloor\frac{x-m+n+l-1}{2}\right\rfloor \left\lfloor\frac{x-n+l+m}{2}\right\rfloor}$

then $\dfrac{1-K}{1+K} = \dfrac{2lmn}{x^2-l^2-m^2-n^2+1 + \dfrac{4(l^2-1^2)(m^2-1^2)(n^2-1^2)}{3(x^2-l^2-m^2-n^2+5)+\&c + \dfrac{4(l^2-2^2)(m^2-2^2)(n^2-2^2)}{5(x^2-l^2-m^2-n^2+9)+\&c}}} = V$

If $K = \dfrac{\left\lfloor\frac{x+l+n-1}{4}\right\rfloor \left\lfloor\frac{x+l-n-3}{4}\right\rfloor \left\lfloor\frac{x-l+n-3}{4}\right\rfloor \left\lfloor\frac{x-l-n-1}{4}\right\rfloor}{\left\lfloor\frac{x-l+n-1}{4}\right\rfloor \left\lfloor\frac{x-l-n-3}{4}\right\rfloor \left\lfloor\frac{x+l-n-1}{4}\right\rfloor \left\lfloor\frac{x+l+n-3}{4}\right\rfloor}$

then $\dfrac{1-K}{1+K} = \dfrac{ln}{x^2-1-l^2 + \dfrac{2^2-n^2}{1+\dfrac{2^2-l^2}{x^2-1+\dfrac{4^2-n^2}{1+\dfrac{4^2-l^2}{x^2-1+\&c}}}}}$

If $\phi(y) = \dfrac{1}{y+1} + \dfrac{1}{y+3} + \dfrac{1}{y+5} + \&c$

then $\phi(x-l-n) - \phi(x+l-n) + \phi(x+l+n) - \phi(x-l+n) = \dfrac{2ln}{(x^2-1)+n^2-l^2 + \dfrac{2(1^2-n^2)}{1+\dfrac{2(1^2-l^2)}{3(x^2-1)+n^2-l^2+\dfrac{4(2^2-n^2)}{1+\dfrac{4(2^2-l^2)}{\&c\ \&c}}}}}$

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2. $\sqrt{1-\frac{1}{x}} = \frac{1-\left(\frac{1}{2}\right)^2\frac{1}{x-1}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{(x-1)^2}-\left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{(x-1)^3}+\&c}{1+\left(\frac{1}{2}\right)^2\frac{1}{x}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{x^2}+\left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{x^3}+\&c}$.

3. $1+\left(\frac{1}{2}\right)^3+\left(\frac{1.3}{2.4}\right)^3+\left(\frac{1.3.5}{2.4.6}\right)^3+\left(\frac{1.3.5.7}{2.4.6.8}\right)^3+\&c$

$= 2\left\{1-\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2-\left(\frac{1.3.5}{2.4.6}\right)^2+\&c\right\}$

$= \left\{1+\left(\frac{1}{2}\right)^2\frac{1}{2}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{2^2}+\left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{2^3}+\&c\right\}^2$.

8. $\frac{1}{nx^n}+\frac{m}{\lfloor 1}\cdot\frac{1}{(n+1)x^{n+1}}+\frac{m(m+1)}{\lfloor 2}\cdot\frac{1}{(n+2)x^{n+2}}+\&c$

$= \frac{1}{n(x-1)^n}+\frac{m-n-1}{\lfloor 1}\cdot\frac{1}{(n+1)(x-1)^{n+1}}+\frac{(m-n-1)(m-n-2)}{\lfloor 2}\cdot\frac{1}{(n+2)(x-1)^{n+2}}+\&c$

9. $\frac{1}{nx^n}+\frac{n}{n(n+1)}\cdot\frac{1}{x^{n+1}}+\frac{n(n+1)}{n(n+1)(n+2)}\cdot\frac{1}{x^{n+2}}+\&c$

$= \frac{1}{n(x-1)^n}-\frac{n}{\lfloor 1}\cdot\frac{1}{(n+1)(x-1)^{n+1}}+\frac{n(n+1)}{\lfloor 2}\cdot\frac{1}{(n+2)(x-1)^{n+2}}+\&c$

10. $\frac{1}{nx}+\frac{\lfloor 1}{n(n+1)x^2}+\frac{\lfloor 2}{n(n+1)(n+2)x^3}+\&c$

$= \frac{1}{n(x-1)}-\frac{1}{(n+1)(x-1)^2}+\frac{1}{(n+2)(x-1)^3}-\&c$

11. $(1-x)^{a+b}\left\{1+\frac{ab}{1.n}x+\frac{a(a+1)b(b+1)}{1.2\,n(n+1)}x^2+\&c\right\}$

$= (1-x)^n\left\{1+\frac{(n-a)(n-b)}{1.n}x+\frac{(n-a)(n-a+1)(n-b)(n-b+1)}{1.2.n(n+1)}x^2+\&c\right.$

Sol. Apply XIII 4 twice.

12. $\frac{\lfloor x+y+n}{\lfloor x+n\ \lfloor y+n}+\frac{PQ}{\lfloor 1}\cdot\frac{\lfloor x+y+n+1}{\lfloor x+n+1\ \lfloor y+n+1}+\frac{P(P-1)\,Q(Q-1)}{\lfloor 2}\cdot\frac{\lfloor x+y+n+2}{\lfloor x+n+2\ \lfloor y+n+2}+\&c.$

$= \frac{\lfloor P+Q+n}{\lfloor P+n\ \lfloor Q+n}+\frac{xy}{\lfloor 1}\cdot\frac{\lfloor P+Q+n+1}{\lfloor P+n+1\ \lfloor Q+n+1}+\frac{x(x-1)\,y(y-1)}{\lfloor 2}\cdot\frac{\lfloor P+Q+n+2}{\lfloor P+n+2\ \lfloor Q+n+2}+\&c.$

Sol. By XIII we have,

$\frac{\lfloor x+y+n}{\lfloor x+n\ \lfloor y+n} = \frac{1}{\lfloor n}+\frac{xy}{\lfloor 1}\cdot\frac{1}{\lfloor n+1}+\frac{x(x-1)\,y(y-1)}{\lfloor 2}\cdot\frac{1}{\lfloor n+2}+\&c$

$\frac{PQ}{\lfloor 1}\cdot\frac{\lfloor x+y+n+1}{\lfloor x+n+1\ \lfloor y+n+1} = \frac{PQ}{\lfloor 1}\cdot\frac{1}{\lfloor n+1}+\frac{PQ}{\lfloor 1}\cdot\frac{xy}{\lfloor 1}\cdot\frac{1}{\lfloor n+2}+\frac{PQ}{\lfloor 1}\cdot\frac{x(x-1)\,y(y-1)}{\lfloor 2}\cdot\frac{1}{\lfloor n+3}+\&c.$

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$$1 - \frac{n}{\lfloor 1} \cdot \frac{1}{2} \cdot \frac{1}{3} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{1.3}{2.4} \cdot \frac{1}{5} - \&c$$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{\lfloor n}{\lfloor n+\frac{1}{2}} \left\{ 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 + \&c \text{ to } n+1 \text{ terms} \right\}$$

$$\frac{\pi}{2} \left\{ \frac{1}{n} + \left(\frac{1}{2}\right)^2 \frac{1}{n+1} + \left(\frac{1.3}{2.4}\right)^2 \frac{1}{n+2} + \&c \right\}$$

$$= \frac{1}{n} + \frac{n+\frac{1}{2}}{n(n+1)} \cdot \frac{1}{3} + \frac{(n+\frac{1}{2})(n+1\frac{1}{2})}{n(n+1)(n+2)} \cdot \frac{1}{5} + \&c$$

$$\frac{1}{n} + \frac{x}{\lfloor 1} \cdot \frac{y}{z} \cdot \frac{1}{n+1} + \frac{x(x-1)}{\lfloor 2} \cdot \frac{y(y-1)}{z(z+1)} \cdot \frac{1}{n+2} + \&c$$

$$= \frac{\lfloor z \, \lfloor n-1}{\lfloor x+n} \left\{ 1 + \frac{x}{\lfloor 1} \cdot \frac{y+z}{z} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{(y+z)(y+z+1)}{z(z+1)} + \&c \right.$$
$$\left. \text{to } x+1 \text{ terms} \right\}$$

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$\frac{P(P-1)\,Q(Q-1)}{\lfloor 2} \frac{\lfloor x+y+n+2}{\lfloor x+n+2 \lfloor y+n+2} = \frac{P(P-1)\,Q(Q-1)}{\lfloor 2} \cdot \frac{1}{\lfloor n+2} + \&c.$

&c &c &c &c &c

Add up all the results.

13. $\frac{1}{P+n} + \frac{x}{1} \cdot \frac{y}{n} \cdot \frac{1}{P+n+1} + \frac{x(x-1)}{\lfloor 2} \cdot \frac{y(y-1)}{n(n+1)} \cdot \frac{1}{P+n+2} + \&c$

$= \frac{\lfloor n-1 \lfloor x+y+n}{\lfloor x+n \lfloor y+n} - P \cdot \frac{\lfloor n-1 \lfloor x+y+n+1}{\lfloor x+n+1 \lfloor y+n+1} + P(P-1) \frac{\lfloor n-1 \lfloor x+y+n+2}{\lfloor x+n+2 \lfloor y+n+2} - \&c$

14. $\frac{\pi}{4} \left\{ \frac{1}{n+1} + \left(\frac{1}{2}\right)^2 \frac{1}{n+2} + \left(\frac{1.3}{2.4}\right)^2 \frac{1}{n+3} + \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{n+4} + \&c \right\}$

$= 1 - \frac{n}{1} \cdot \left(\frac{2}{3}\right)^2 + \frac{n(n-1)}{\lfloor 2} \cdot \left(\frac{2.4}{3.5}\right)^2 - \frac{n(n-1)(n-2)}{\lfloor 3} \cdot \left(\frac{2.4.6}{3.5.7}\right)^2 + \&c.$

$= \left(\frac{2.4.6.\ldots\ldots 2n}{3.5.7.\ldots\ldots 2n+1}\right)^2 \left\{ 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 + \&c \text{ to } n+1 \text{ terms} \right\}$

Cor1. $\frac{\pi}{4} \left\{ 1 + \left(\frac{1}{2}\right)^2 \frac{1}{3} + \left(\frac{1.3}{2.4}\right)^2 \frac{1}{5} + \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{1}{7} + \&c \right\}$

$= \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c$

2. $\pi n \left\{ \frac{1}{n} + \left(\frac{1}{2}\right)^2 \frac{1}{n+1} + \left(\frac{1.3}{2.4}\right)^2 \frac{1}{n+2} + \&c \right\} - \log n$

is finite when $n = \infty$.

15. $\frac{1}{y+n} - \frac{x}{n} \cdot \frac{1}{y+n+1} + \frac{x(x-1)}{n(n+1)} \cdot \frac{1}{y+n+2} - \&c$

$= \frac{1}{x+n} - \frac{y}{n} \cdot \frac{1}{x+n+1} + \frac{y(y-1)}{n(n+1)} \cdot \frac{1}{x+n+2} - \&c$

16. $\frac{1}{\{\phi(x)\}^n} \left[1 + \frac{n}{1} \cdot \frac{m}{2m} \left\{ \frac{\phi(x) - \phi(-x)}{\phi(x)} \right\} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{m(m+1)}{2m(2m+1)} \left\{ \frac{\phi(x) - \phi(-x)}{\phi(x)} \right\}^2 + \&c \right]$

is always an even function of x.

17. $1 + \frac{n}{1} \cdot \frac{m}{2m} \cdot \left(\frac{2x}{1+x}\right) + \frac{n(n+1)}{\lfloor 2} \cdot \frac{m(m+1)}{2m(2m+1)} \left(\frac{2x}{1+x}\right)^2 + \&c$

$= (1+x)^n \left\{ 1 + \frac{n(n+1)}{2(2m+1)} x^2 + \frac{n(n+1)(n+2)(n+3)}{2.4.(2m+1)(2m+3)} x^4 + \&c \right\}$

18. $1 + \frac{n}{1} \cdot \frac{m}{2m} \cdot \frac{4x}{(1+x)^2} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{m(m+1)}{2m(2m+1)} \left\{ \frac{4x}{(1+x)^2} \right\}^2 + \&c$

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$$\phi(0) + \frac{2\phi'(0)}{\lfloor 1} \cdot \frac{m}{2m} + \frac{2^2\phi''(0)}{\lfloor 2} \cdot \frac{m.(m+1)}{2m(2m+1)} + \&$$

$$= \phi(1) + \frac{\phi''(1)}{2(2m+1)} + \frac{\phi^{IV}(1)}{2.4.(2m+1)(2m+3)} + \&c$$

If $m(m-1) = 2p$

$$e^{-mx}\left\{1 + \frac{1}{2}\cdot\frac{m}{\lfloor 1}(1-e^{-2x}) + \frac{1.3}{2.4}\cdot\frac{m(m+1)}{\lfloor 2}(1-e^{-2x})^2 + \&\right\}$$

$$= 1 + \frac{A_1 x^2}{2(\lfloor 1)^2} + \frac{A_2 x^4}{2^2(\lfloor 2)^2} + \frac{A_3 x^6}{2^3(\lfloor 3)^2} + \&c$$

where $A_n = p^n - \frac{n(n-1)}{\lfloor 3} p^{n-1} + \frac{n(n-1)(n-2)(3n-1)}{\lfloor 5} p^{n-2}$

$$- \ldots \&c + (-1)^{n-1} 2p \cdot \frac{\lfloor n-1}{1.3.5\ldots(2n-1)} (2^{2n}-1) B_{2n} = \phi(p)$$

$$\phi(1) = \frac{\lfloor n}{1.3.5\ldots(2n-1)}$$

$$\left\{1 + \left(\tfrac{x}{2}\right)^2 x + \quad\right\}^2 = \cfrac{1}{1 - \cfrac{x}{2 - \cfrac{3x}{8 - \cfrac{5x}{2 - \cfrac{17x}{40 - \cfrac{23x}{2 - \cfrac{1395x}{3128 - 4x}}}}}}}$$

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$= (1+x)^{2n}\left\{1 + \frac{n}{\lfloor 1}\cdot\frac{n-m+\frac{1}{2}}{m+\frac{1}{2}}x^2 + \frac{n(n+1)}{\lfloor 2}\cdot\frac{(n-m+\frac{1}{2})(n-m+1\frac{1}{2})}{(m+\frac{1}{2})(m+1\frac{1}{2})}x^4 + \&c\right\}$

19. $1 + \frac{2x}{\lfloor 1}\cdot\frac{m}{2m} + \frac{(2x)^2}{\lfloor 2}\cdot\frac{m(m+1)}{2m(2m+1)} + \frac{(2x)^3}{\lfloor 3}\cdot\frac{m(m+1)(m+2)}{2m(2m+1)(2m+2)}$ &c

$= e^x\left\{1 + \frac{x^2}{2}\cdot\frac{1}{2m+1} + \frac{x^4}{\lfloor 4}\cdot\frac{1}{(2m+1)(2m+3)} + \&c\right\}$

Cor. $1 + \frac{1}{2}\cdot\frac{x}{\lfloor 1} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^2}{\lfloor 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^3}{\lfloor 3} + \&c$

$= e^{\frac{x}{2}}\left\{1 + \frac{x^2}{4^2} + \frac{x^4}{4^2\cdot 8^2} + \frac{x^6}{4^2\cdot 8^2\cdot 12^2} + \&c\right\}$

Ex. 1. $1 - \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi^2}{\lfloor 2} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{\pi^4}{\lfloor 4} - \&c = 0.$

2. $1 - \frac{1\cdot 3}{1^2\cdot 2^2}\pi^2 + \frac{1\cdot 3\cdot 5\cdot 7}{1^2\cdot 2^2\cdot 3^2\cdot 4^2}\pi^4 - \&c$

$= -\left(1 - \frac{\pi^2}{2^2} + \frac{\pi^4}{2^2\cdot 4^2} - \frac{\pi^6}{2^2\cdot 4^2\cdot 6^2} + \&c\right)$

3. $1 - \frac{1\cdot 3}{3\cdot 6}\cdot\frac{\pi^2}{\lfloor 2} + \frac{1\cdot 3\cdot 5\cdot 7}{3\cdot 6\cdot 9\cdot 12}\cdot\frac{\pi^4}{\lfloor 4} - \&c$

$= \frac{1}{2}\left(1 - \frac{\pi^2}{6^2} + \frac{\pi^4}{6^2\cdot 12^2} - \frac{\pi^6}{6^2\cdot 12^2\cdot 18^2} + \&c\right).$

20. i. $1 + \left(\frac{1}{2}\right)^2\frac{4x}{(1+x)^2} + \left\{\frac{1\cdot 3}{2\cdot 4}\left(\frac{4x}{(1+x)^2}\right)^2\right\}^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left\{\frac{4x}{(1+x)^2}\right\}^3 + \&c$

$= (1+x)\left\{1 + \left(\frac{1}{2}\right)^2x^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2x^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2x^6 + \&c\right\}$

ii. $1 + \left(\frac{1}{2}\right)^2\frac{2x}{1+x} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{2x}{1+x}\right)^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(\frac{2x}{1+x}\right)^3 + \&c$

$= \sqrt{1+x}\left(1 + \frac{1\cdot 3}{4^2}x^2 + \frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}x^4 + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{4^2\cdot 8^2\cdot 12^2}x^6 + \&c\right\}$

Ex. 1. $1 + \frac{n}{\lfloor 1}\cdot\frac{m}{2m}\cdot\frac{4x}{(1+x)^2} + \frac{n(n+1)}{\lfloor 2}\cdot\frac{m(m+1)}{2m(2m+1)}\left\{\frac{4x}{(1+x)^2}\right\}^2 + \&c$

$= \frac{(1-x^4)^{2m}}{(1-x)^{2n}}\left\{1 + \frac{2m-n}{\lfloor 1}\cdot\frac{m-n+\frac{1}{2}}{m+\frac{1}{2}}x^2 + \frac{(2m-n)(2m-n+1)}{\lfloor 2}\times\right.$

$\left.\frac{(m-n+\frac{1}{2})(m-n+1\frac{1}{2})}{(m+\frac{1}{2})(m+1\frac{1}{2})}x^4 + \&c\right\}$

Sol. Apply XIII ii in R. H. S of XIII 18.

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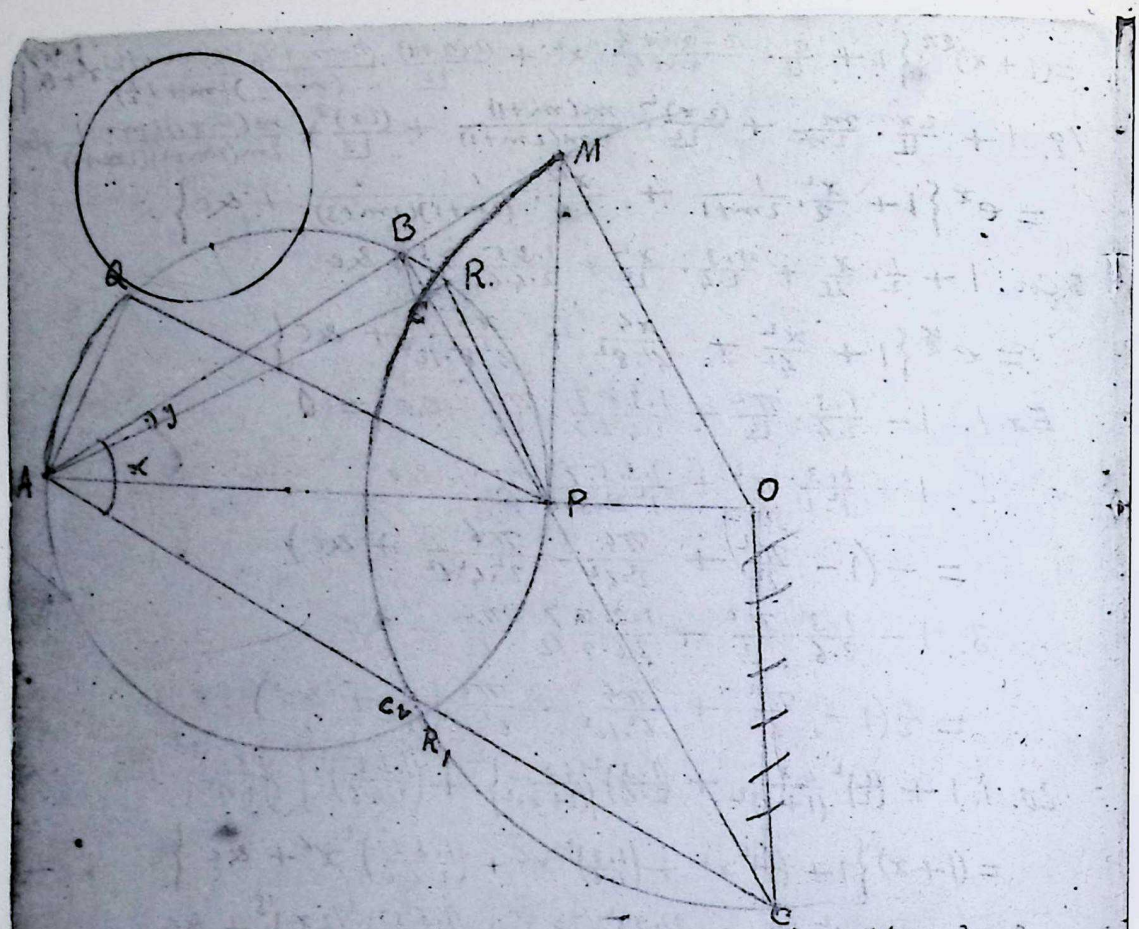

$$\frac{\frac{\beta}{\gamma}x + \frac{\alpha}{1\!\!1}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha-1)}{\lfloor 2}\cdot\frac{\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 + \&c}{1 + \frac{\alpha}{1\!\!1}\cdot\frac{\beta}{\gamma}x + \frac{\alpha(\alpha+1)}{\lfloor 2}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \&c}$$

$$= \frac{\beta x}{\gamma - (\alpha+\beta+1)x} + \frac{(\beta+1)(\alpha+\gamma+1)x}{\gamma+1-(\alpha+\beta+2)x} + \frac{(\beta+2)(\alpha+\gamma+2)x}{\gamma+2-(\alpha+\beta+3)x} + \&c$$

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2. $1+\frac{n}{\lfloor 1}\cdot\frac{m}{\lfloor m}\cdot\frac{2x}{1+x}+\frac{n(n+1)}{\lfloor 2}\cdot\frac{m(m+1)}{2m(2m+1)}\left(\frac{2x}{1+x}\right)^2+\&c$

$=\frac{(1-x^2)^m}{(1-x)^n}\left\{1+\frac{(2m-n)(2m-n+1)}{2.(2m+1)}x^2+\frac{(2m-n)(2m-n+1)}{2.4}\times\right.$

$\left.\frac{(2m-n+2)(2m-n+3)}{(2m+1)(2m+3)}x^4+\&c\right\}$

Sol. Apply XIII 11 in R.H.S of XIII 17.

3. $1+\frac{n(n+1)}{2(2m+1)}\cdot\frac{4x}{(1+x)^2}+\frac{n(n+1)(n+2)(n+3)}{2.4(2m+1)(2m+3)}\left\{\frac{4x}{(1+x)^2}\right\}^2+\&c$

$=(1+x)^n\left\{1+\frac{n}{\lfloor 1}\cdot\frac{n-m+\frac{1}{2}}{m+\frac{1}{2}}x+\frac{n(n+1)}{\lfloor 2}\cdot\frac{(n-m+\frac{1}{2})(n-m+1\frac{1}{2})}{(m+\frac{1}{2})(m+1\frac{1}{2})}x^2+\&c\right\}$

Sol. Combine the results of XIII 17 & 18.

4. $1+\frac{n(n+1)}{1^2}\cdot\frac{x}{(1+x)^2}+\frac{n(n+1)(n+2)(n+3)}{1^2.2^2}\cdot\frac{x^2}{(1+x)^4}+\&c$

$=(1+x)^n\left\{1+\frac{n^2}{(\lfloor 1)^2}x+\frac{n^2(n+1)^2}{(\lfloor 2)^2}x^2+\frac{n^2(n+1)^2(n+2)^2}{(\lfloor 3)^2}x^3+\&c\right\}$

21. $\left\{1+\frac{x}{\lfloor 1}\cdot\frac{1}{m}+\frac{x^2}{\lfloor 2}\cdot\frac{1}{m(m+1)}+\frac{x^3}{\lfloor 3}\cdot\frac{1}{m(m+1)(m+2)}+\&c\right\}$

$\times\left\{1+\frac{x}{\lfloor 1}\cdot\frac{1}{n}+\frac{x^2}{\lfloor 2}\cdot\frac{1}{n(n+1)}+\frac{x^3}{\lfloor 3}\cdot\frac{1}{n(n+1)(n+2)}+\&c\right\}$

$=1+\frac{x}{\lfloor 1}\cdot\frac{m+n}{mn}+\frac{x^2}{\lfloor 2}\cdot\frac{(m+n+1)(m+n+2)}{m(m+1)n(n+1)}+$

$\frac{x^3}{\lfloor 3}\cdot\frac{(m+n+2)(m+n+3)(m+n+4)}{m(m+1)(m+2)n(n+1)(n+2)}+\&c$

Sol. From we have,

$\frac{\lfloor m\,x^t}{\lfloor m+t\,\lfloor t}+\frac{1}{\lfloor 1(n+1)}\cdot\frac{x^t\lfloor m}{\lfloor m+t-1\,\lfloor t-1}+\frac{1}{\lfloor 1}\cdot\frac{x^t}{(n+1)(n+t)}\cdot\frac{\lfloor m}{\lfloor m+t-2\,\lfloor t-2}$

$=\frac{\lfloor n\,\lfloor m+2t+n\,\lfloor m\,x^t}{\lfloor m+t+n\,\lfloor t+n\,\lfloor m+t\,\lfloor t}$.

Equate the series whose tth terms are these.

In a similar manner XIII 22, 23, 24, 25 & 26 may be proved.

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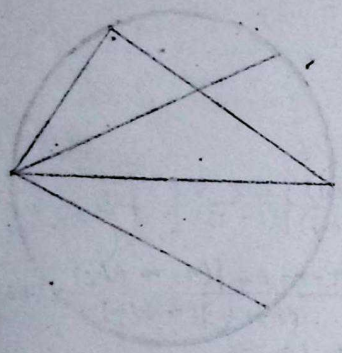
$$\frac{1}{1+\dfrac{a_1x}{1+\dfrac{a_2x}{1+\dfrac{a_3x}{1+\&}}}} = 1 - A_1x + A_2x^2 - A_3x^3 + \&$$



Let $P_n = a_1 a_2 \cdots a_{n-1}(a_1 + a_2 + \cdots a_n)$

$P_1 = A_1$

$P_2 = A_2$

$P_3 = A_3 - a_1 A_2$

$P_4 = A_4 - (a_1 + a_2) A_3$

$P_5 = A_5 - (a_1 + a_2 + a_3) A_4 + a_1 a_3 A_3$

$P_6 = A_6 - (a_1 + a_2 + a_3 + a_4) A_5 + (a_1 a_3 + a_1 a_4 + a_2 a_4) A_4$

$P_n = \phi_0(n) A_n - \phi_1(n) A_{n-1} + \phi_2(n) A_{n-2} - \&c$

where $\phi_n(n+1) - \phi_n(n) = a_{n-1} \phi_{n-1}(n-1)$.

$$\frac{1}{1+b_1x+\dfrac{a_1x}{1+b_2x+\dfrac{a_2x}{1+b_3x+\&c}}} = 1 - A_1x + A_2x^2 - A_3x^3 + \&c$$

$$P_n = a_1 a_2 a_3 \cdots a_{n-1}\left(\frac{1}{a_1+b_1} + \frac{1}{a_2+b_2} + \cdots + \frac{1}{a_n+b_n}\right)$$

$$= \phi_0(n) A_n - \phi_1(n) A_{n-1} + \phi_2(n) A_{n-2} - \&c$$

where $\phi_n(n+1) - \phi_n(n) = b_n \phi_{n-1}(n) + a_{n-1} \phi_{n-1}(n-1)$.

$D_{n-1} = \phi_0(n) + x\phi_1(n) + x^2\phi_2(n) + x^3\phi_3(n) + \&c$

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Cor. $e^x\left\{1+\frac{x^2}{2}\cdot\frac{1}{2n+1}+\frac{x^4}{2.4}\cdot\frac{1}{(2n+1)(2n+3)}+\&c\right\}$

$=1+\frac{2x}{\lfloor 1}\cdot\frac{n}{2n}+\frac{(2x)^2}{\lfloor 2}\cdot\frac{n(n+1)}{2n(2n+1)}+\&c$

2. $\left\{1+\frac{x}{4}\cdot\frac{1}{2n+1}+\frac{x^2}{4.8}\cdot\frac{1}{(2n+1)(2n+3)}+\&c\right\}^2$

$=1+\frac{x}{\lfloor 1}\cdot\frac{n}{2n}\cdot\frac{1}{2n+1}+\frac{x^2}{\lfloor 2}\cdot\frac{n(n+1)}{2n(2n+1)}\cdot\frac{1}{(2n+1)(2n+3)}+\&c$

Ex. 1. $1+\frac{1}{2}\cdot\frac{x}{\lfloor 1}+\frac{1.3}{2.4}\cdot\frac{x^2}{\lfloor 2}+\frac{1.3.5}{2.4.6}\cdot\frac{x^3}{\lfloor 3}+\&c$

$=e^{\frac{x}{2}}\left(1+\frac{x^2}{4^2}+\frac{x^4}{4^4\lfloor 2^2}+\frac{x^6}{4^6\lfloor 3^2}+\&c\right)$.

2. $1+\frac{1}{2}\cdot\frac{x}{(\lfloor 1)^2}+\frac{1.3}{2.4}\cdot\frac{x^2}{(\lfloor 2)^2}+\frac{1.3.5}{2.4.6}\cdot\frac{x^3}{(\lfloor 3)^2}+\&c$

$=\left(1+\frac{x}{\lfloor 2}+\frac{x^2}{2^2\lfloor 2^2}+\frac{x^4}{2^4\lfloor 3^2}+\frac{x^6}{2^6\lfloor 4^2\lfloor 3^2}+\&c\right)^2$.

22. $\left\{1+\frac{x}{\lfloor 1}\cdot\frac{1}{m+1}\cdot\frac{1}{n+1}+\frac{x^2}{\lfloor 2}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n+1)(n+2)}+\&c\right\}$

$\times\left\{1-\frac{x}{\lfloor 1}\cdot\frac{1}{m+1}\cdot\frac{1}{n+1}+\frac{x^2}{\lfloor 2}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n+1)(n+2)}-\&c\right\}$

$=1-\frac{x^2}{\lfloor 1}\cdot\frac{m+n+3}{(m+1)(n+1)}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n+1)(n+2)}$

$+\frac{x^4}{\lfloor 2}\frac{(m+n+5)(m+n+6)}{(m+1)(m+2)(n+1)(n+2)}\cdot\frac{1}{(m+1)(m+2)(m+3)(m+4)}$

$\times\frac{1}{(n+1)(n+2)(n+3)(n+4)}-\frac{x^6}{\lfloor 3}\frac{(m+n+7)(m+n+8)(m+n+9)}{(m+1)(m+2)(m+3)(n+1)(n+2)(n+3)}$.

$\times\frac{1}{(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)}\times\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)}$

$+\&c$

23. $\left\{1+\frac{x}{\lfloor 1}\frac{1}{(m+n+1)}\frac{1}{n+1}+\frac{x^2}{\lfloor 2}\frac{1}{(m+n+1)(m+n+2)}\frac{1}{(n+1)(n+2)}+\&c\right\}$

$\left\{1+\frac{x}{\lfloor 1}\cdot\frac{1}{m+1}\cdot\frac{1}{n-1}+\frac{x^2}{\lfloor 2}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n-1)(n-2)}+\&c\right\}$

$=1+\frac{x}{\lfloor 1}\cdot\frac{2m+n+3}{m+n+1}\cdot\frac{1}{m+1}\cdot\frac{n}{n^2-1^2}+\frac{x^2}{\lfloor 2}\frac{(2m+n+4)(2m+n+5)}{(m+n+1)(m+n+2)}$

$\times\frac{1}{(m+1)(m+2)}\cdot\frac{1}{n^2-2^2}\cdot+$

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$$n\left\{1+\frac{x^n}{\lfloor n}+\frac{x^{2n}}{\lfloor 2n}+\frac{x^{3n}}{\lfloor 3n}+\&\right\}$$

$$= e^x + e^{x\cos\frac{2\pi}{n}}\cos\left(x\sin\frac{2\pi}{n}\right) + e^{x\cos\frac{4\pi}{n}}\cos\left(x\sin\frac{4\pi}{n}\right)$$

$$+ \text{upto } n \text{ terms}.$$

$$\frac{x^4}{\lfloor 4} + \left(\frac{x^{4+n}}{\lfloor 4+n} + \frac{x^{4-n}}{\lfloor 4-n}\right) + \left(\frac{x^{4+2n}}{\lfloor 4+2n} + \frac{x^{4-2n}}{\lfloor 4-2n}\right) + \&$$

$$= 1 + \left(\frac{x^n}{\lfloor n} + \frac{x^{-n}}{\lfloor -n}\right) + \left(\frac{x^{2n}}{\lfloor 2n} + \frac{x^{-2n}}{\lfloor -2n}\right) + \&c$$

$$\left\{6m^2 + (3m^3 - m)\right\}^3 + \left\{6m^2 - (3m^3 - m)\right\}^3$$

$$= \left\{6m^2(3m^2+1)\right\}^2.$$

$$\left\{m^7 - 3m^4(1+p) + m\left(3\sqrt{1+p}^2 - 1\right)\right\}^3$$

$$+ \left\{2m^6 - 3m^3(1+2p) + (1+3p+3p^4)\right\}^3$$

$$+ \left\{m^6 - (1+3p+3p^2)\right\}^3.$$

$$= \left\{m^7 - 3m^4 p + m(3p^2 - 1)\right\}^3$$

$$\int_{-\infty}^{\infty} \frac{\phi(x)}{\lfloor x}\,dx = \phi(\alpha) + \frac{\phi'(\alpha)}{\lfloor 1} + \frac{\phi''(\alpha)}{\lfloor 2} + \frac{\phi'''(\alpha)}{\lfloor 3} + \&c$$

$$\int_{-\infty}^{\infty} \frac{a^x}{\lfloor x}\,dx = e^a \quad \int_{-\infty}^{\infty} \frac{\lfloor n}{\lfloor x\,\lfloor n-x}\,a^x\,dx = (1+a)^n.$$

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$$+\frac{x^3}{\lfloor 8}\frac{(2m+n+5)(2m+n+7)(2m+n+9)}{(m+n+1)(m+n+2)(m+n+3)}\frac{1}{(m+1)(m+2)(m+3)}\times$$

$$\frac{n}{(n^2-1^2)(n^2-3^2)}+\frac{x^4}{\lfloor 4}\frac{(2m+n+6)(2m+n+8)(2m+n+10)(2m+n+12)}{(m+n+1)(m+n+2)(m+n+3)(m+n+4)}$$

$$\times\frac{1}{(m+1)(m+2)(m+3)(m+4)}\frac{1}{(n^2-2^2)(n^2-4^2)}+\&c$$

24. $\left\{1+\frac{x}{\lfloor 1}\cdot\frac{m}{n+1}+\frac{x^2}{\lfloor 2}\cdot\frac{m(m-1)}{(n+1)(n+2)}+\&c\right\}$

$\times\left\{1-\frac{x}{\lfloor 1}\cdot\frac{m}{n+1}+\frac{x^2}{\lfloor 2}\cdot\frac{m(m-1)}{(n+1)(n+2)}-\&c\right\}$

$=1-\frac{x^2}{\lfloor 1}\frac{m+n+1}{(n+1)(n+2)}\cdot\frac{m}{n+1}+\frac{x^4}{\lfloor 2}\cdot\frac{(m+n+1)(m+n+2)}{(n+1)(n+2)(n+3)(n+4)}$

$\times\frac{m(m-1)}{(n+1)(n+2)}-\&c$

25. $\left\{1+\frac{x}{\lfloor 1}mn+\frac{x^2}{\lfloor 2}m(m-1)n(n-1)+\&c\right\}$

$\times\left\{1-\frac{x}{\lfloor 1}mn+\frac{x^2}{\lfloor 2}m(m-1)n(n-1)-\&c\right\}$

$=1-\frac{x^2}{\lfloor 1}mn(m+n-1)+\frac{x^4}{\lfloor 2}m(m-1)n(n-1)(m+n-2)(m+n-3)$

$-\frac{x^6}{\lfloor 3}m(m-1)(m-2)n(n-1)(n-2)(m+n-3)(m+n-4)(m+n-5)+\&$

26. $\left\{1+\frac{x}{\lfloor 1}\cdot\frac{m}{n+1}+\frac{x^2}{\lfloor 2}\frac{m(m+1)}{(n+1)(n+2)}+\&c\right\}$

$\times\left\{1+\frac{x}{\lfloor 1}\cdot\frac{m+n}{n-1}+\frac{x^2}{\lfloor 2}\cdot\frac{(m+n)(m+n-1)}{(n-1)(n-2)}+\&c\right\}$

$=1+\frac{x}{\lfloor 1}(2m+n+1)\frac{n}{n^2-1^2}+\frac{x^2}{\lfloor 2}(2m+n)(2m+n+2)\frac{1}{n^2-2^2}$

$+\frac{x^3}{\lfloor 3}(2m+n-1)(2m+n+1)(2m+n+3)\cdot\frac{n}{(n^2-1^2)(n^2-3^2)}+\&$

Ex. 1. $\left(1+\frac{x^3}{\lfloor 3}+\frac{x^6}{\lfloor 6}+\frac{x^9}{\lfloor 9}+\&c\right)\left(1-\frac{x^3}{\lfloor 3}+\frac{x^6}{\lfloor 6}-\frac{x^9}{\lfloor 9}+\&c\right)$

$=\frac{1}{3}+\frac{2}{3}\left\{1-\frac{(3x^4)^3}{\lfloor 6}+\frac{(3x^4)^6}{\lfloor 12}-\frac{(3x^4)^9}{\lfloor 18}+\&c\right\}$

2. $\left\{1+\frac{x}{(\lfloor 1)^3}+\frac{x^2}{(\lfloor 2)^3}+\frac{x^3}{(\lfloor 3)^3}+\&c\right\}\left\{1-\frac{x}{(\lfloor 1)^3}+\frac{x^2}{(\lfloor 2)^3}-\&c\right\}$

$=1-\frac{\lfloor 3}{(\lfloor 1\,\lfloor 2)^3}x^2+\frac{\lfloor 6}{(\lfloor 2\,\lfloor 4)^3}x^4-\frac{\lfloor 9}{(\lfloor 3\,\lfloor 6)^3}x^6+\&c.$

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$$1 + \frac{1+x}{\lfloor 1} \cdot \frac{mn}{m+n+1} + \frac{(1+x)^2}{\lfloor 2} \cdot \frac{m(m+1)\,n(n+1)}{(m+n+1)(m+n+2)} + \&c$$

$$= \sqrt{\pi}\, \frac{\lfloor \frac{m+n-1}{2}}{\lfloor \frac{m-1}{2} \lfloor \frac{n-1}{2}} \left\{ 1 + \frac{x^2}{\lfloor 2} mn + \frac{x^4}{\lfloor 4} m(m+2)\, n(n+2) + \&c \right\}$$

$$+ 2\sqrt{\pi}\, \frac{\lfloor \frac{m+n-1}{2}}{\lfloor \frac{m}{2} - 1 \lfloor \frac{n}{2} - 1} \left\{ \frac{x}{\lfloor 1} + \frac{x^3}{\lfloor 3} (m+1)(n+1) + \frac{x^5}{\lfloor 5} \cdot (m+1)(m+3)(n+1)(n+3) + \&c \right\}$$

I : α, II : β, IV : γ then

$$\frac{\sqrt{3}\, \sqrt[6]{\beta(1-\beta)}}{\sqrt[3]{\alpha(1-\gamma)} - \sqrt[3]{\gamma(1-\alpha)}} = \frac{\left\{ 1 + \frac{1.2}{3^2} \alpha + \&c \right\} \left\{ 1 + \frac{1.2}{3^2} \gamma + \&c \right\}}{\left\{ 1 + \frac{1.2}{3^2} \beta + \&c \right\}^2}$$

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3. $\left(x+\frac{x^4}{\lfloor 4}+\frac{x^7}{\lfloor 7}+\frac{x^{10}}{\lfloor 10}+\&c\right)\left(x-\frac{x^4}{\lfloor 4}+\frac{x^7}{\lfloor 7}-\frac{x^{10}}{\lfloor 10}+\&c\right)$

$=\frac{2}{3}\left\{\frac{(3x^2)}{\lfloor 2}-\frac{(3x^2)^4}{\lfloor 8}+\frac{(3x^2)^7}{\lfloor 14}-\frac{(3x^2)^{10}}{\lfloor 20}+\&c\right\}.$

4. $\cos x\cosh x=1-\frac{(2x^2)^2}{\lfloor 4}+\frac{(2x^2)^4}{\lfloor 8}-\frac{(2x^2)^6}{\lfloor 12}+\&c$

5. $\sin x\sinh x=\frac{(2x^2)}{\lfloor 2}-\frac{(2x^2)^3}{\lfloor 6}+\frac{(2x^2)^5}{\lfloor 10}-\&c$

6. $\left\{1+\frac{x}{(\lfloor 1)^2}+\frac{x^2}{(\lfloor 2)^2}+\frac{x^3}{(\lfloor 3)^2}+\&c\right\}\left\{1-\frac{x}{(\lfloor 1)^2}+\frac{x^2}{(\lfloor 2)^2}-\frac{x^3}{(\lfloor 3)^2}+\&c\right\}$

$=1-\frac{x^2}{\lfloor 2}\cdot\frac{1}{(\lfloor 1)^2}+\frac{x^4}{\lfloor 4}\cdot\frac{1}{(\lfloor 2)^2}-\frac{x^6}{\lfloor 6}\cdot\frac{1}{(\lfloor 3)^2}+\&c$

7. $\left\{1+\frac{1}{2}\cdot\frac{x}{\lfloor 1}+\frac{1.3}{2.4}\cdot\frac{x^2}{\lfloor 2}+\frac{1.3.5}{2.4.6}\cdot\frac{x^3}{\lfloor 3}+\&c\right\}\left\{1-\frac{1}{2}\cdot\frac{x}{\lfloor 1}+\frac{1.3}{2.4}\cdot\frac{x^2}{\lfloor 2}\ \&c\right\}$

$=1+\frac{1}{2}\cdot\frac{x^2}{2^2}+\frac{1.3}{2.4}\cdot\frac{x^4}{2^2.4^2}+\frac{1.3.5}{2.4.6}\cdot\frac{x^6}{2^2.4^2.6^2}+\&c$

8. $\left\{\frac{1}{n}+\frac{x}{n(n+1)}+\frac{x^2}{n(n+1)(n+2)}+\frac{x^3}{n(n+1)(n+2)(n+3)}+\&c\right\}$

$\times\left\{\frac{1}{n}-\frac{x}{n(n+1)}+\frac{x^2}{n(n+1)(n+2)}-\frac{x^3}{n(n+1)(n+2)(n+3)}+\&c\right\}$

$=\frac{1}{n}\cdot\frac{1}{n}+\frac{x^2}{n(n+1)(n+2)}\cdot\frac{1}{n+1}+\frac{x^4}{n(n+1)(n+2)(n+3)(n+4)}\cdot\frac{1}{n+2}+\&c$

9. $\left(1+\frac{x}{1.3}+\frac{x^2}{1.3.5}+\frac{x^3}{1.3.5.7}+\&c\right)\left(1-\frac{x}{1.3}+\frac{x^2}{1.3.5}\cdot\frac{x^3}{1.3.7}+\&c\right)$

$=1+\frac{x^2}{1.3.5}\cdot\frac{1}{3}+\frac{x^4}{1.3.5.7.9}\cdot\frac{1}{5}+\&c$

10. $\left\{1+xn+x^2n(n-1)+x^3n(n-1)(n-2)+\&c\right\}$

$\times\left\{1-xn+x^2n(n-1)-x^3n(n-1)(n-2)+\&c\right\}$

$=\frac{n}{n}+\frac{x^2}{n-1}n(n-1)(n-2)+\frac{x^4}{n-2}n(n-1)(n-2)(n-3)(n-4)+\&c$

27. $1+\frac{x}{\lfloor 1}\cdot\frac{y}{x+y+1}+\frac{x(x+1)}{\lfloor 2}\cdot\frac{y(y+1)}{(x+y+1)(x+y+3)}+$

$\frac{x(x+1)(x+2)}{\lfloor 3}\cdot\frac{y(y+1)(y+2)}{(x+y+1)(x+y+3)(x+y+5)}+\&c=\frac{\lfloor\frac{x+y-1}{2}}{\lfloor\frac{x-1}{2}\lfloor\frac{y-1}{2}}\sqrt{\pi}$

Cor. 1. $\frac{\lfloor\frac{n-1}{2}}{\lfloor\frac{n+x-1}{2}\lfloor\frac{n-x-1}{2}}\sqrt{\frac{2\pi}{2^n}}=1+\frac{1^2-x^2}{4(n+1)}+\frac{(1^2-x^2)(3^2-x^2)}{4.8.(n+1)(n+3)}$

$+\frac{(1^2-x^2)(3^2-x^2)(5^2-x^2)}{4.8.12.(n+1)(n+3)(n+5)}-+\&c$

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$$\int_0^\infty \frac{x^{n-1}}{1+x}\left\{1-\frac{\alpha\beta}{(\alpha+\beta)\lfloor 1}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)\lfloor 2}x^2-\&c\right\}dx$$

$$=\frac{\lfloor\alpha-n\ \lfloor\beta-n\ \lfloor n-1}{\lfloor\alpha+\beta-n-1}\left\{\frac{1}{\alpha+\beta-n}+\frac{\alpha\beta}{(\alpha+\beta)\lfloor 1}\cdot\frac{1}{\alpha+\beta-n+1}\right.$$

$$\left.+\frac{\alpha(\alpha+1)\beta(\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)\lfloor 2}\cdot\frac{1}{\alpha+\beta-n+2}+\&c\right\}$$

$$=\frac{\lfloor\alpha-n\ \lfloor n-1}{\lfloor\alpha-1}\left\{\frac{1}{\alpha}+\frac{\alpha n}{(\alpha+\beta)\lfloor 1}\cdot\frac{1}{\alpha+1}+\frac{\alpha(\alpha+1)n(n+1)}{(\alpha+\beta)(\alpha+\beta+1)\lfloor 2}\right.$$

$$\left.\times\frac{1}{\alpha+2}+\&c\right\}.$$

If $\alpha+\beta=1$, then

$$1+\frac{\alpha}{\lfloor 1}\cdot\frac{\beta}{\gamma+1}\cdot\left(\frac{1-\sqrt{1-x}}{2}\right)+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\lfloor 2(\gamma+1)(\gamma+2)}\cdot\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c$$

$$\overline{1+\frac{\alpha+\gamma}{4}\cdot\frac{\beta+\gamma}{\gamma+1}x+\frac{(\alpha+\gamma)(\alpha+\gamma+2)(\beta+\gamma)(\beta+\gamma+1)}{4\cdot 8\,(\gamma+1)(\gamma+2)}x^2+\&c}$$

$$=\left(\frac{1+\sqrt{1-x}}{2}\right)^{\gamma}.$$

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2. $\frac{\lfloor\frac{n-1}{2}}{\left(\lfloor\frac{n-1}{n}\right)^2}\sqrt{\frac{2\pi}{2^n}} = 1 + \frac{1^2}{4(n+1)} + \frac{1^2 \cdot 3^2}{4 \cdot 8 \cdot (n+1)(n+3)} + \&c$ 104

3. $\frac{\lfloor\frac{n-1}{2}}{\lfloor\frac{2n-3}{8}\lfloor\frac{2n-5}{8}}\sqrt{\frac{2\pi}{2^n}} = 1 + \frac{1.3}{16(n+1)} + \frac{1.3.5.7}{16.32(n+1)(n+3)} + \&c$

28. If $x+y+z=0$, then

$\frac{1}{n} + \frac{x}{\lfloor 1} \cdot \frac{y}{z} \cdot \frac{1}{n+1} + \frac{x(x-1)}{\lfloor 2} \cdot \frac{y(y-1)}{z(z+1)} \cdot \frac{1}{n+2} + \&c$

$= \frac{\lfloor n-1 \lfloor x+y+n}{\lfloor x+n \lfloor y+n} \left\{ 1 + \frac{x}{\lfloor 1} \cdot \frac{y}{z} + \frac{x(x-1)}{\lfloor 2} \cdot \frac{y(y-1)}{z(z+1)} + \&c \text{ to } x+y+n+1 \text{ terms} \right\}$

29. If $x+y+z = \frac{1}{2}$, then

$1 + \frac{x}{\lfloor 1} \cdot \frac{y}{z} P + \frac{x(x-1)}{\lfloor 2} \cdot \frac{y(y-1)}{z(z+1)} P^2 + \frac{x(x-1)(x-2)}{\lfloor 3} \cdot \frac{y(y-1)(y-2)}{z(z+1)(z+2)} P^3 + \&c$

$= 1 + \frac{2x}{\lfloor 1} \cdot \frac{2y}{z} \cdot \left(\frac{1-\sqrt{1-P}}{2}\right) + \frac{2x(2x-1)}{\lfloor 2} \cdot \frac{2y(2y-1)}{z(z+1)} \cdot \left(\frac{1-\sqrt{1-P}}{2}\right)^2 + \&c$

Cor. $1 + \frac{1^2+n}{4^2} x + \frac{(1^2+n)(3^2+n)}{4^2 \cdot 8^2} x^2 + \frac{(1^2+n)(3^2+n)(5^2+n)}{4^2 \cdot 8^2 \cdot 12^2} x^3 + \&c$

$= 1 + \frac{1^2+n}{2^2}\left(\frac{1-\sqrt{1-x}}{2}\right) + \frac{(1^2+n)(3^2+n)}{2^2 \cdot 4^2} \cdot \left(\frac{1-\sqrt{1-x}}{2}\right)^2 + \&c$

30. If $x+y+z=0$, then

$\left\{ 1 + \frac{x}{\lfloor 1} \cdot \frac{y}{x+\frac{1}{2}} P + \frac{x(x-1)}{\lfloor 2} \cdot \frac{y(y-1)}{(x+\frac{1}{2})(x+1\frac{1}{2})} P^2 + \&c \right\}^2$

$= 1 + \frac{2x}{\lfloor 1} \cdot \frac{2y}{z+\frac{1}{2}} \cdot \frac{z}{2z} P + \frac{2x(2x-1)}{\lfloor 2} \cdot \frac{2y(2y-1)}{(z+\frac{1}{2})(z+1\frac{1}{2})} P^2 + \&c$

Cor 1. $\left\{ 1 + \frac{1^2+n}{4^2} x + \frac{(1^2+n)(3^2+n)}{4^2 \cdot 8^2} x^2 + \&c \right\}^2$

$= 1 + \frac{1}{2} \cdot \frac{1^2+n}{2^2} x + \frac{1.3}{2.4} \cdot \frac{(1^2+n)(3^2+n)}{2^2 \cdot 4^2} x^2 + \&c$

& $\left(1 + \frac{x}{2^2} + \frac{x^2}{2^2 \cdot 4^2} + \frac{x^3}{2^2 \cdot 4^2 \cdot 6^2} + \&c\right)^2$

$= 1 + \frac{1}{2} \cdot \frac{x}{(\lfloor 1)^2} + \frac{1.3}{2.4} \cdot \frac{x^2}{(\lfloor 2)^2} + \frac{1.3.5}{2.4.6} \cdot \frac{x^3}{(\lfloor 3)^2} + \&c$

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If p th & q th be $\phi(x)$ & $\psi(x)$ and n th be $f(x)$
then if p th & q th be $\phi F(x)$ & $\psi F(x)$ then n th $= f F(x)$.
(ii) if p th & q th be $F\phi(x)$ & $F\psi(x)$ then n th $= Ff(x)$.

Thus we may add or subtract any constant
(& multiply or divide by any constant)
to x in each function or to each function

$I = x \quad II = x^2 + 2x$ then n th $= (x+1)^n - 1$

$I = x \quad II = x^2 + 4x$ then n th $= \left\{\left(\frac{\sqrt{x+4} + \sqrt{x}}{2}\right)^n - \left(\frac{\sqrt{x+4} - \sqrt{x}}{2}\right)^n\right\}$

$I = x \quad II = x^2 - 2$ then n th $= \left(\frac{x + \sqrt{x^2-4}}{2}\right)^n + \left(\frac{x - \sqrt{x^2-4}}{2}\right)^n$

$I = x \quad II = \frac{x^2}{(1-x)^2}$ n th $= \frac{4x^n}{\left\{(1+\sqrt{1-x})^n + (1-\sqrt{1-x})^n\right\}^2}$

If $I = x$ & II $x^2 + 2nx$, then

$III = x^3 + 3nx^2 + 3 \cdot \frac{n(n+1)}{2} x - \frac{n(n-1)(n-2)x}{2x} + \frac{3(n+1)}{2}$.

If $\chi(x)$ y be the common equation then the
required sum $= \frac{y}{x(1-x)\frac{\partial y}{\partial x}}$ or any power of
this or more generally if $f(x)$ & $F(x)$ be
of the p th & q th degree find $\phi(x)$ such
that $\sqrt[p]{\phi f(x)} = \sqrt[q]{\phi F(x)} = \chi(x)$, then

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CHAPTER XIV

1. $\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots+\frac{a_n}{b_n}$. $= a_1\frac{N_{n-1}}{D_n} = \frac{a_1}{D_0D_1} - \frac{a_1a_2}{D_1D_2} + \frac{a_2a_2a_3}{D_2D_3} - \&c$ to n terms.

$N_{n-1} = b_nN_{n-2} + a_nN_{n-3}$ and $D_n = b_nD_{n-1} + a_nD_{n-2}$.

Cor. $a_1+a_2+a_3+\&c$ to n terms $= \frac{a_1}{1-}\frac{a_2}{a_1+a_2-}\frac{a_1a_3}{a_2+a_3-}\frac{a_2a_4}{a_3+a_4-}\frac{a_3a_5}{a_4+a_5-}\&c$ to n terms.

2. $x = x-a_1 + \frac{xa_1}{x-a_2+}\frac{xa_2}{x-a_3+}\frac{xa_3}{x-a_4+}\&c$

3. $x = a_1 + \sqrt{x^2+a_1(a_1+2a_2)} - 2a_1\sqrt{x^2+a_2(a_2+2a_3)} - 2a_2\sqrt{\&c\,\&c}$

4. $x+n+a = \sqrt{ax+(n+a)^2} + x\sqrt{a(x+n)+(n+a)^2} + (x+n)\sqrt{\&c}$

Ex. 1. $3 = 1\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\&c}}}}$

2. $4 = 1\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\&c}}}}$

5. $\frac{1}{x+a} - \frac{1}{(x+a)(x+2a)} + \frac{1}{(x+a)(x+2a)(x+3a)} - \&c$

$= \frac{1}{x+a+}\frac{x+a}{x+2a-1+}\frac{x+2a}{x+3a-1+}\frac{x+3a}{x+4a-1+\&c}$

Cor. $\frac{1}{e-1} = \frac{1}{1+}\frac{2}{2+}\frac{3}{3+\&c}$

6. $x = \frac{x+1}{x+}\frac{x+2}{x+1+}\frac{x+3}{x+2+\&c}$

Cor. $1 = \frac{2}{1+}\frac{3}{2+}\frac{4}{3+\&c}$

7. $\frac{x+a+1}{x+1} = \frac{x+a}{x-1+}\frac{x+2a}{x+a-1+}\frac{x+3a}{x+2a-1+\&c}$

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the function for the nth degree $= \phi^{-1}\{\chi(x)\}^n$.

The self-repeating series $\sqrt[n]{\frac{\phi(x)}{\psi(x)\phi'(x)}}$ where n is any quantity and $\psi(x)$ any known function.

supposing the series to be $S(x)$. then

$$\frac{SF(x)}{Sf(x)} = \sqrt[n]{\frac{p}{q} \cdot \frac{\psi f(x) F'(x)}{\psi F(x) f'(x)}}$$

If $y = e^{-2\pi} \cdot \frac{1 + \frac{1\cdot5}{6^2}(1-x) + \&c}{1 + \frac{1\cdot5}{6^2}x + \&c}$

then $1 - 504\left(\frac{1^5 y}{1-y} + \frac{2^5 y^2}{1-y^2} + \frac{3^5 y^3}{1-y^3} + \&c\right)$

$= \left\{1 + \frac{1\cdot5}{6^2}x + \&c.\right\}^6 (1-2x)$

If $y = e^{-\frac{2\pi}{\sqrt{3}}} \cdot \frac{1 + \frac{1\cdot2}{3^2}(1-x) + \&c}{1 + \frac{1\cdot2}{3^2}x + \&c}$

then $1 + 240\left(\frac{1^3 y}{1-y} + \frac{2^3 y^2}{1-y^2} + \frac{3^3 y^3}{1-y^3} + \&c\right)$

$= \left\{1 + \frac{1\cdot2}{3^2}x + \&c\right\}^4 (1+8x)$.

$1 - 504\left(\frac{1^5 y}{1-y} + \frac{2^5 y^2}{1-y^2} + \frac{3^5 y^3}{1-y^3} + \&c\right)$

$= \left\{1 + \frac{1\cdot2}{3^2}x + \&c\right\}^6 (1 - 20x - 8x^2)$

If $y = e^{-\pi\sqrt{2}} \cdot \frac{1 + \frac{1\cdot3}{4^2}(1-x) + \&c}{1 + \frac{1\cdot3}{4^2}x + \&c}$. then

$1 + 240\left(\frac{1^3 y}{1-y} + \frac{2^3 y^2}{1-y^2} + \&c\right) = \left(1 + \frac{1\cdot3}{4^2}x + \&c\right)^4 (1+3x)$.

$1 - 504\left(\frac{1^5 y}{1-y} + \frac{2^5 y^2}{1-y^2} + \&c\right) = \left(1 + \frac{1\cdot3}{4^2}x + \&c\right)^6 (1-9x)$.

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Cor. 1. $\frac{4}{3} = \frac{3}{1+\frac{4}{2+\frac{5}{3+\frac{6}{4+}}}}$ &c

2. $\frac{5}{3} = \frac{4}{1+\frac{6}{3+\frac{8}{5+\frac{10}{7+}}}}$ &c

8. If x is a positive integer,

$$x = \frac{1}{1-x+\frac{2}{2-x+\frac{3}{3-x+\cdots\cdots+\frac{x}{0+\frac{x+1}{1+\frac{x+2}{2+}}}}}} \text{ &c}$$

9 If a is a positive integer and if

$$\frac{Na}{Da} = \frac{n}{n-a+\frac{n+1}{n-a+1+\frac{n+2}{n-a+2+}}} \text{ &c, then}$$

$$\frac{Na+1}{Na} = n+2-a+\frac{a-1}{n+3-a+\frac{a-2}{n+4-a+}} \text{ &c}$$

Here we should equate the numerators and the denominators (in lowest terms)

If $N = \phi(n)$, then $D = \phi(n+1)$.

Cor 1. $\frac{n^2+n+1}{n^2-n+1} = \frac{n}{n-3+\frac{n+1}{n-2+\frac{n+2}{n-1+}}}$ &c

2. $\frac{n^3+2n+1}{(n-1)^3+2(n-1)+1} = \frac{n}{n-4+\frac{n+1}{n-3+\frac{n+2}{n-2+}}}$ &c

10. $1 = \frac{x+a}{a+\frac{(x+a)^2-a^2}{a+\frac{(x+2a)^2-a^2}{a+}}}$ &c

11. If $a < b$, then

$$a = \frac{ab}{a+b+d} - \frac{(a+d)(b+d)}{a+b+3d} - \frac{(a+2d)(b+2d)}{a+b+5d} -$$

12. $\frac{a_1}{x+\frac{a_2}{1+\frac{a_3}{x+\frac{a_4}{1+}}}}$ &c to 2n terms

$$= \frac{a_1}{x+a_2} - \frac{a_2 a_3}{x+a_3+a_4} - \frac{a_4 a_5}{x+a_5+a_6} - \text{ &c to } n \text{ terms}$$

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$$\sqrt{2}\left\{\frac{1}{2}+e^{-\frac{\pi x}{x^2+y^2}}\cos\left(\frac{\pi y}{x^2+y^2}\right)+e^{-\frac{4\pi x}{x^2+y^2}}\cos\left(\frac{4\pi y}{x^2+y^2}\right)+\&c\right\}$$

$$=\sqrt{\sqrt{x^2+y^2}+x}\left\{\frac{1}{2}+e^{-\pi x}\cos\pi y+e^{-4\pi x}\cos 4\pi y+\&c\right\}$$

$$+\sqrt{\sqrt{x^2+y^2}-x}\left\{e^{-\pi x}\sin\pi y+e^{-4\pi x}\sin 4\pi y+\&c\right\}$$

$$\sqrt{2}\left\{e^{-\frac{\pi x}{x^2+y^2}}\sin\left(\frac{\pi y}{x^2+y^2}\right)+e^{-\frac{4\pi x}{x^2+y^2}}\sin\left(\frac{4\pi y}{x^2+y^2}\right)+\&c\right\}$$

$$=\sqrt{\sqrt{x^2+y^2}-x}\left\{\frac{1}{2}+e^{-\pi x}\cos\pi y+e^{-4\pi x}\cos 4\pi y+\&c\right\}$$

$$-\sqrt{\sqrt{x^2+y^2}+x}\left\{e^{-\pi x}\sin\pi y+e^{-4\pi x}\sin 4\pi y+\&c\right\}$$

$$\frac{1}{\pi}-\frac{4}{5^2}+\frac{4}{5^2}-\&c=.915\,965\,594\,177$$

If $\alpha=\frac{27p(1+p)^4}{2(1+4p+p^2)^3}$ & $\beta=\frac{27p^4(1+p)}{2(2+2p-p^2)^3}$ then

$$\left(1+p-\frac{p^2}{2}\right)\left\{1+\frac{1.2}{3^2}\alpha+\frac{1.2.4.5}{3^2.6^2}\alpha^2+\&c\right\}$$

$$=(1+4p+p^2)\left\{1+\frac{1.2}{3^2}\beta+\frac{1.2.4.5}{3^2.6^2}\beta^2+\&c\right\}$$

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13. $\dfrac{a_1+h}{1+\dfrac{a_1}{x+}\dfrac{a_2+h}{1+\dfrac{a_2}{x+}\dfrac{a_3+h}{1+}}}$ &c

$= h + \dfrac{a_1}{1+\dfrac{a_1+h}{x+}\dfrac{a_2}{1+}\dfrac{a_2+h}{x+}}$ &c

14. $\dfrac{1}{(m+1)(n+1)} - \dfrac{1}{(m+2)(n+2)}$ &c

$= \dfrac{1}{m+n+1+mn} + \dfrac{(m+1)^2(n+1)^2}{m+n+3+}\dfrac{(m+2)^2(n+2)^2}{m+n+5+}$ &c

15. $\dfrac{a_1 x}{b_1+}\dfrac{a_2 x}{b_2+}\dfrac{a_3 x}{b_3+}$ &c $= T_1 x - T_2 x^2 + T_3 x^3 -$ &c

Let $\dfrac{P_n}{b_n} = \dfrac{a_1 a_2 \cdots a_n}{(b_1 b_2 \cdots b_n)^2}$ and $T_n - P_n = t_n$, then

$t_1 = 0$, $t_2 = 0$

$T_1 t_3 - T_2^2 = 0$

$T_2 t_4 - T_3^2 = 0$

$T_3 t_5 - T_4^2 = \dfrac{M^2}{P_1 P_3}$ where $M = T_2 T_4 - T_3^2$

$T_4 t_6 - T_5^2 = \dfrac{N^2}{P_2 P_4}$ where $N = T_3 T_5 - T_4^2$

16. $\dfrac{(x+1)^n - (x-1)^n}{(x+1)^n + (x-1)^n} = \dfrac{n}{x+}\dfrac{n^2-1^2}{3x+}\dfrac{n^2-2^2}{5x+}\dfrac{n^2-3^2}{7x+}$ &c $= A_n$ $= \dfrac{2}{A_{2n}}$

$A_n + \dfrac{1}{A_n}$

Cor. 1. $\tan\dfrac{1}{x} = \dfrac{1}{x+}\dfrac{1^2}{3x+}\dfrac{2^2}{5x+}\dfrac{3^2}{7x+}$ &c

2. $\log_e \dfrac{x+1}{x-1} = \dfrac{2}{x-}\dfrac{1^2}{3x-}\dfrac{2^2}{5x-}\dfrac{3^2}{7x-}$ &c

3. $\tan\dfrac{1}{x} = \dfrac{1}{x-}\dfrac{1}{3x-}\dfrac{1}{5x-}\dfrac{1}{7x-}$ &c

4. $\dfrac{e^x-1}{e^x+1} = \dfrac{x}{2+}\dfrac{x^2}{6+}\dfrac{x^2}{10+}\dfrac{x^2}{14+}$ &c

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$$e^{-\pi\sqrt{2}\cdot\frac{1+\frac{1\cdot3}{4^2}(1-\alpha)}{1+\frac{1\cdot3}{4^2}\alpha}} = y$$

III degree $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 4\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)} = 1$

VII degree $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 20\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}$
$+ 8\sqrt{2}\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\left\{\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right\} = 1$

$$\left\{1 + \frac{1\cdot3}{4^2}x + \frac{1\cdot3\cdot5\cdot7}{4^2\cdot8^2}x^2 + \&c\right\}^2$$
$$= 1 + 24\left(\frac{y}{1-y} + \frac{3y^3}{1-y^3} + \frac{5y^5}{1-y^5} + \&c\right)$$

V degree $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 8\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}\left(\sqrt[6]{\alpha\beta} + \sqrt[6]{(1-\alpha)(1-\beta)}\right)$
$= 1$

$$F\left(\frac{3\sqrt{2}-\sqrt{5}-2}{3\sqrt{2}+\sqrt{5}+2}\right) = e^{-\pi\sqrt{10}} = F\left\{(\sqrt{10}-3)^2(\sqrt{2}-1)^4\right\}.$$

$$F\left(\frac{7\sqrt{2}-2\sqrt{6}-5}{7\sqrt{2}+2\sqrt{6}+5}\right) = e^{-3\pi\sqrt{2}}$$

$$1 + \frac{1\cdot2}{3^2}\left\{1 - \left(\frac{1-t}{1+2t}\right)^3\right\} + \&c$$
$$= (1+2t)\left\{1 + \frac{1\cdot2}{3^2}t^3 + \frac{1\cdot2\cdot4\cdot5}{3^2\cdot6^2}t^6 + \&c\right\}$$

XI $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 16\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}\left(\sqrt[3]{\alpha\beta} + \sqrt[3]{1-\alpha.1-\beta}\right)$
$+ 48\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}\left(\sqrt[6]{\alpha\beta} + \sqrt[6]{(1-\alpha)(1-\beta)}\right)$
$+ 68\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)} = 1$

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17. $$\frac{\frac{x}{n}+\frac{x^2}{\lfloor 2}\cdot\frac{1}{n(n+1)}+\frac{x^3}{\lfloor 3}\cdot\frac{1}{n(n+1)(n+2)}+\&c}{1+\frac{x}{\lfloor 1}\cdot\frac{1}{n}+\frac{x^2}{\lfloor 2}\cdot\frac{1}{n(n+1)}+\&c}$$

$$=\frac{x}{n}+\frac{x}{n+1}+\frac{x}{n+2}+\frac{x}{n+3}+\&c$$

Sol: Let $\phi(x)=\frac{x}{\lfloor 0 \lfloor 1}+\frac{x^2}{\lfloor 1 \lfloor 2}+\frac{x^3}{\lfloor 2 \lfloor 3}+\frac{x^4}{\lfloor 3 \lfloor 4}+\&c$, then we see that

$\phi(x)=x\phi''(x)$

$\therefore \phi'(x)=\phi''(x)+x\phi'''(x)$

$\therefore \phi''(x)=2\phi'''(x)+x\phi^{IV}(x)$

$\therefore \phi'''(x)=3\phi^{IV}(x)+x\phi^{V}(x)$

&c &c

$$\therefore \frac{\phi(x)}{\phi'(x)}=\frac{x\phi''(x)}{\phi''(x)+x\phi'''(x)}=\frac{x}{1+x\frac{\phi'''(x)}{\phi''(x)}}=\frac{x}{1+\frac{x\phi'''(x)}{2\phi'''(x)+x\phi^{IV}(x)}}$$

$$=\frac{x}{1+\frac{x}{2+x\frac{\phi^{IV}(x)}{\phi'''(x)}}}$$

$$=\frac{x}{1+\frac{x}{2+\frac{x}{3+\cdots+\frac{x}{n-1}+\frac{\frac{x}{n}+\frac{x^2}{\lfloor 2}\cdot\frac{1}{n(n+1)}+\frac{x^3}{\lfloor 3}\cdot\frac{1}{n(n+1)(n+2)}+\&c}{1+\frac{x}{\lfloor 1}\cdot\frac{1}{n}+\frac{x^2}{\lfloor 2}\cdot\frac{1}{n(n+1)}+\&c}}}}$$

18. $$\frac{\frac{m}{n}x+\frac{n-m}{\lfloor 1}\cdot\frac{m(m+1)}{n(n+1)}x^2+\frac{(n-m)(n-m-1)}{\lfloor 2}\cdot\frac{m(m+1)(m+2)}{n(n+1)(n+2)}x^3+\&c}{1+\frac{n-m}{\lfloor 1}\cdot\frac{m}{n}x+\frac{(n-m)(n-m+1)}{\lfloor 2}\cdot\frac{m(m+1)}{n(n+1)}x^2+\&c}$$

$$=\frac{xmn}{n+\frac{x(m-n)(n-n)}{n+1+\frac{x(m+1)(n+1)}{n+2+\frac{x(m+1)(m-n-1)(n-n-1)}{n+3+\&c}}}}$$

19. $$\frac{n}{m}x-\frac{n(n+1)}{m(m+1)}x^2+\&c$$

$$=\frac{nx}{m+\frac{n(n+1)x}{n+1+\frac{1(m-n)x}{n+2+\frac{(n+1)(n+2)x}{n+3+\&c}}}}$$

$$=\frac{nx}{m+\frac{1(1+x)}{n+\frac{(n+2)x}{1+\frac{1(n+1)x(1+x)}{n+1+x(n+3)}-\frac{2(n+1)2(1+x)}{n+2+x(n+4)}}}}$$

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If $y = e^{-2\pi} \cdot \dfrac{1 + \frac{1\cdot5}{6^2}(1-x) + \frac{1\cdot5\cdot7\cdot11}{6^2\cdot12^2}(1-x)^2 + \&c}{1 + \frac{1\cdot5}{6^2}x + \frac{1\cdot5\cdot7\cdot11}{6^2\cdot12^2}x^2 + \&c}$

then $\left\{1 + \frac{1\cdot5}{6^2}x + \frac{1\cdot5\cdot7\cdot11}{6^2\cdot12^2}x^2 + \&c\right\}^4$

$= 1 + 240\left(\frac{1^3y}{1-y} + \frac{2^3y^2}{1-y^2} + \frac{3^3y^3}{1-y^3} + \&c\right)$

If $1 + \frac{1\cdot5}{6^2}x + \frac{1\cdot5\cdot7\cdot11}{6^2\cdot12^2}x^2 + \&c$

$= \left\{1 + \left(\frac{1}{2}\right)^2 y + \left(\frac{1\cdot3}{2\cdot4}\right)^2 y^2 + \&c\right\}\sqrt{1-y+y^2}$

& $x(1-x) = A$ & $y(1-y) = B$

then $A = \dfrac{27B^2}{16(1-B)^3}$

If $y = \dfrac{p(2+p)}{1+2p}$ then $x = \dfrac{27}{4}\cdot\dfrac{(p+p^2)^2}{(1+p+p^2)^3}$

If $z = \dfrac{p^3(2+p)}{1+2p}$ and $x = \dfrac{27}{4}\cdot\dfrac{(p+p^2)^2}{(1+p+p^2)^3}$

then $1 + \frac{1\cdot2}{3^2}x + \frac{1\cdot2\cdot4\cdot5}{3^2\cdot6^2}x^2 + \&c$

$= \left\{1 + \left(\frac{1}{2}\right)^2 z + \left(\frac{1\cdot3}{2\cdot4}\right)^2 z^2 + \&c\right\}\cdot\dfrac{1+p+p^2}{\sqrt{1+2p}}$

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Sol. Write $n+1$ for n in XIV 18. Subtract both sides from 1 and equate the reciprocals of the result.

Cor 1. $\frac{x}{m} + \frac{x^2}{m(m+1)} + \frac{x^3}{m(m+1)(m+2)} + \&c$

$= \frac{x}{m-} \frac{mx}{m+1} + \frac{1x}{m+2} - \frac{(m+1)x}{m+3} + \frac{2x}{m+4} - \&c$

$= \frac{x}{m-x} + \frac{x}{m+1-x} + \frac{2x}{m+2-x} + \frac{3x}{m+3-x} + \&c$

Cor 2. $1 + \frac{x}{x+1} + \frac{x^2}{(x+1)(x+2)} + \&c$

$= 1 + \frac{2x}{2} + \frac{3x}{3} + \frac{4x}{4} + \&c$

20. $\frac{\left\lfloor\frac{x+m+n-1}{2}\right\rfloor \left\lfloor\frac{x-m-n-1}{2}\right\rfloor - \left\lfloor\frac{x+m-n-1}{2}\right\rfloor \left\lfloor\frac{x-m+n-1}{2}\right\rfloor}{\left\lfloor\frac{x+m+n-1}{2}\right\rfloor \left\lfloor\frac{x-m-n-1}{2}\right\rfloor + \left\lfloor\frac{x+m-n-1}{2}\right\rfloor \left\lfloor\frac{x-m+n-1}{2}\right\rfloor}$

$= \frac{mx}{x + \frac{(m^2-1^2)(n^2-1^2)}{3x + \frac{(m^2-2^2)(n^2-2^2)}{5x + \frac{(m^2-3^2)(n^2-3^2)}{7x + \&c}}}}$

21. $\frac{\left\lfloor\frac{x+n-3}{4}\right\rfloor \left\lfloor\frac{x-n-3}{4}\right\rfloor}{\left\lfloor\frac{x+n-1}{4}\right\rfloor \left\lfloor\frac{x-n-1}{4}\right\rfloor} = \frac{4}{x - \frac{n^2-1^2}{2x - \frac{n^2-3^2}{2x - \frac{n^2-5^2}{2x - \&c}}}}$

Cor 1. $\left(\frac{\left\lfloor\frac{x-3}{4}\right\rfloor}{\left\lfloor\frac{x-1}{4}\right\rfloor}\right)^2 = \frac{4}{x + \frac{1^2}{2x + \frac{3^2}{2x + \frac{5^2}{2x + \&c}}}}$

Cor 2. $\frac{\left\lfloor\frac{x-5}{8}\right\rfloor \left\lfloor\frac{x-7}{8}\right\rfloor}{\left\lfloor\frac{x-1}{8}\right\rfloor \left\lfloor\frac{x-3}{8}\right\rfloor} = \frac{8}{x + \frac{1\cdot 3}{2x + \frac{5\cdot 7}{2x + \frac{9\cdot 11}{2x + \&c}}}}$

22. $\left\{\frac{\left\lfloor\frac{x+n-3}{4}\right\rfloor \left\lfloor\frac{x-n-3}{4}\right\rfloor}{\left\lfloor\frac{x+n-1}{4}\right\rfloor \left\lfloor\frac{x-n-1}{4}\right\rfloor}\right\}^2 = \frac{8}{\frac{x^2+n^2-1}{2} + \frac{1^2 n^2}{1 + \frac{x^2-1}{2} + \frac{2^2 n^2}{1 + \frac{x^2-2^2}{2} + \&c}}}$

Cor $\left\{\frac{\left\lfloor\frac{x-3}{4}\right\rfloor}{\left\lfloor\frac{x-1}{4}\right\rfloor}\right\}^4 = \frac{8}{x^2-1}$

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$$\alpha = \frac{b(3+b)^2}{2(1+b)^3} \qquad \beta = \frac{b^2(3+b)}{4}$$

$$1-\alpha = \frac{(1-b)^2(2+b)}{2(1+b)^3} \qquad 1-\beta = \frac{(1-b)(2+b)^2}{4}$$

$$1 + \frac{1\cdot2}{3^2}\alpha + \frac{1\cdot2\cdot4\cdot5}{3^2\cdot6^2}\alpha^2 + \frac{1\cdot2\cdot4\cdot5\cdot7\cdot8}{3^2\cdot6^2\cdot9^2}\alpha^3 + \dots$$

$$= (1+b)\left\{1 + \frac{1\cdot2}{3^2}\beta + \frac{1\cdot2\cdot4\cdot5}{3^2\cdot6^2}\beta^2 + \frac{1\cdot2\cdot4\cdot5\cdot7\cdot8}{3^2\cdot6^2\cdot9^2}\beta^3 + \dots\right\}$$

II dep. $\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} = 1$

$$\sqrt[3]{\frac{\alpha^2}{\beta}} - \sqrt[3]{\frac{(1-\alpha)^2}{1-\beta}} = \frac{2}{1+b}$$

$$\sqrt[3]{\frac{(1-\beta)^2}{1-\alpha}} - \sqrt[3]{\frac{\beta^2}{\alpha}} = 1+b$$

$$\sqrt[3]{\frac{\alpha^2}{\beta}} + \sqrt[3]{\frac{(1-\alpha)^2}{1-\beta}} = \frac{4}{(1+b)^2}$$

$$\sqrt[3]{\frac{(1-\beta)^2}{1-\alpha}} + \sqrt[3]{\frac{\beta^2}{\alpha}} = (1+b)^2$$

7 dep. $\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} + 3\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)} = 1$

XI. $\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} + 3\sqrt{3}\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}\left\{\sqrt[6]{\alpha\beta} + \sqrt[6]{(1-\alpha)(1-\beta)}\right\} + 6\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)} = 1.$

1, 2, 4, 8

$\sqrt[3]{\alpha\delta} + \sqrt[3]{(1-\alpha)(1-\delta)} + 3\sqrt[6]{\beta\gamma(1-\beta)(1-\gamma)}$ ×

$1 + \frac{1\cdot2}{3\cdot4}\beta + \dots$, $1 + \frac{4\cdot2}{3\cdot4}\gamma + \dots$

$1 + \frac{1\cdot2}{3\cdot4}\alpha + \dots$ $1 - \frac{1\cdot2}{3\cdot4}\delta + \dots$ $= 1$

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23. $\left(\frac{1}{x+n+1} - \frac{1}{x+n+3} + \frac{1}{x+n+5} - \frac{1}{x+n+7} + \&c\right)$
$+\left(\frac{1}{x-n+1} - \frac{1}{x-n+3} + \frac{1}{x-n+5} - \frac{1}{x-n+7} + \&c\right)$
$= \frac{1}{x} + \frac{1^2-n^2}{x} + \frac{2^2}{x} + \frac{3^2-n^2}{x} + \frac{4^2}{x} + \&c$

Cor. $2\left(\frac{1}{x+1} - \frac{1}{x+3} + \frac{1}{x+5} - \frac{1}{x+7} + \&c\right) = \frac{1}{x} + \frac{1^2}{x} + \frac{2^2}{x} + \frac{3^2}{x} + \&c$

24. $\left(\frac{1}{x-n+1} + \frac{1}{x-n+3} + \frac{1}{x-n+5} + \&c\right)$
$-\left(\frac{1}{x+n+1} + \frac{1}{x+n+3} + \frac{1}{x+n+5} + \&c\right)$
$= \frac{n}{x} + \frac{1^2(1^2-n^2)}{3x} + \frac{2^2(2^2-n^2)}{5x} + \frac{3^2(3^2-n^2)}{7x} + \&c$

Cor. $2\left\{\frac{1}{(x+1)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+5)^2} + \&c\right\}$
$= \frac{1}{x} + \frac{1^4}{3x} + \frac{2^4}{5x} + \frac{3^4}{7x} + \&c$

25. $2x^2\left\{\frac{1}{2x^2} - \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} - \frac{1}{(x+3)^2} + \&c\right\}$
$= \frac{1}{x} + \frac{1^2}{x} + \frac{1.2}{x} + \frac{2^2}{x} + \frac{2.3}{x} + \frac{3^2}{x} + \&c$

26. $2x\left(\frac{1}{2x} - \frac{1}{x+2} + \frac{1}{x+4} - \frac{1}{x+6} + \&c\right)$
$= \frac{1}{x} + \frac{1.2}{x} + \frac{2.3}{x} + \frac{3.4}{x} + \&c$

$\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3}$
$= \frac{1}{2x^2+2x} + \frac{1^3}{1+} \quad \frac{1^3}{6x^2+6x+} \quad \frac{2^3}{10x^2+10x+} \quad \frac{2^6}{5(x^2+2x+5)}$

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$$y = e^{-\frac{2\pi}{\sqrt{3}}} \cdot \frac{1 + \frac{1.2}{3^2}(-x) + \frac{1.2.4.5}{3^4.6^2}(-x)^2 + \alpha}{1 + \frac{1.2}{3^2}x + \frac{1.2.4.5}{3^4.6^2}x^2 + \alpha}$$

$$\text{Num } 1 + \frac{1.2}{3^2}x + \frac{1.2.4.5}{3^4.6^2}x^2 + \alpha$$

$$= 1 + 6\left(\frac{y}{1-y} - \frac{y^2}{1-y^2} + \frac{y^4}{1-y^4} - \frac{y^5}{1-y^5} + \alpha\right)$$

$$1 + 12\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \alpha\right)$$

$$= \left\{1 + 6\left(\frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} - \frac{x^5}{1-x^5} + \alpha\right)\right\}^2$$

$$1 + 4\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \frac{6x^6}{1-x^6} + \frac{8x^8}{1-x^8}\right.$$

$$= \left\{1 + 2\left(\frac{x}{1-x} + \frac{x^2}{1-x^2} - \frac{x^3}{1-x^3} + \frac{x^4}{1-x^4} - \frac{x^5}{1-x^5} - \frac{x^6}{1-x^6}\right.\right.$$

$$\left.\left. + \frac{x^8}{1-x^8} + \frac{x^9}{1-x^9} - \frac{x^{10}}{1-x^{10}} + \frac{x^{11}}{1-x^{11}} - \alpha\right)\right\}^2$$

$$1 + 6\left(\frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} - \alpha\right)$$

$$= 4\frac{\psi^3(x^4)}{\psi(x^6)} - 3\frac{\phi^3(x^3)}{\phi(x)}$$

$$1 + 6\left(\frac{1}{e^{4\pi}-1} - \frac{1}{e^{2\pi}-1} + \frac{1}{e^{4\pi}-1} - \alpha\right)$$

$$= \frac{\phi^3(e^{3\pi})}{\phi(e^{2\pi})}(1 + 4p + p^2)$$

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Cor. $\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \&c$

$= \frac{1}{2(x^2+x)+1} + \frac{1}{24(x^2+x)^3 + 60(x^2+x)^2 + \frac{78}{5}(x^2+x) + \frac{\phi(\infty)}{40}}$

$\phi(0) = 198$, $\phi(\frac{1}{2}) = 571$, $\phi(1) = 1015$, $\phi(1\frac{1}{2}) = 1384$, $\phi(2) = 1679$,
$\phi(2\frac{1}{2}) = 1916$, $\phi(3) = 2093$ nearly and $\phi(\infty) = 2880$.

If h is a positive proper fraction, then

$$\frac{\phi(2+h) - \phi(2)}{\phi(3) - \phi(2)} = \frac{3h\,\phi(2)}{2\phi(3) + \frac{h}{2}\{3\phi(2) - 2\phi(3)\}} \text{ nearly.}$$

28. A series of the form $A_0 + (A_1 + A_{-1}) + (A_2 + A_{-2}) + \&c$ is called a perfect series. Hence we see that the series $A_0 + A_1 + A_2 + A_3 + \&c$ is only perfect when A_{-1}, A_{-2}, A_{-3} are all equal to 0. Thus $1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \&c$ is perfect.

If $\phi(0) + \phi(1) + \phi(2) + \phi(3) + \&c$ is a perfect series, then

$$\left.\begin{aligned} &\phi(x) + \{\phi(x+1) + \phi(x-1)\} + \{\phi(x+2) + \phi(x-2)\} + \&c \\ &= \phi(y) + \{\phi(y+1) + \phi(y-1)\} + \{\phi(y+2) + \phi(y-2)\} + \&c \end{aligned}\right\} \text{for all values of } x \text{ and } y$$

29. $A_0 + A_1 + A_2 + \ldots\ldots + A_n$

$= (A_n + A_{n-1} + A_{n-2} + \&c$ ad inf.)

$-(A_{-1} + A_{-2} + A_{-3} + \&c$ ad inf.) for all values of n.

N. B. We also know that $A_0 + A_1 + \ldots A_n$ for all values of n

$= (A_0 - A_{n+1}) + (A_1 - A_{n+2}) + (A_2 - A_{n+3}) + \&c$ if $A_\infty = 0$

Cor. $\phi(0) + \frac{n}{\lfloor 1}\phi(1) + \frac{n(n-1)}{\lfloor 2}\phi(2) + \&c$ ad inf.

$= \phi(n) + \frac{n}{\lfloor 1}\phi(n-1) + \frac{n(n-1)}{\lfloor 2}\phi(n-2) + \&c$ ad inf.

Ex. If either or both of m & x be very great shew that

$$1 + \frac{m}{n+1}x + \frac{m(m-1)}{(n+1)(n+2)}x^2 + \&c = \frac{\lfloor m \;\lfloor n}{x^n \lfloor m+n}(1+x)^{m+\theta x}$$

where θ is less than $\frac{n}{x(m+1)}$ and shew that for

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$$x\psi^3(x)\psi(x^5) - 5x^2\psi(x)\psi^3(x^5)$$

$$= \frac{x}{1-x^2} - \frac{2x^2}{1-x^4} - \frac{3x^3}{1-x^6} + \frac{4x^4}{1-x^8} + \frac{6x^6}{1-x^{12}}$$

$$5\phi(x)\phi^3(x^5) - \phi^2(x)\phi(x^5)$$

$$= 4\left\{1 + \frac{x}{1+x} - \frac{2x^2}{1-x^2} - \frac{3x^3}{1+x^3} + \frac{4x^4}{1-x^4} + \frac{6x^6}{1-x^6} - \frac{7x^7}{1+x^7} - \frac{8x^8}{1-x^8} + \frac{9x^9}{1+x^9} + \cdots\right\}$$

$$25\phi(x)\phi^3(x^5) - \frac{\phi^5(x)}{\phi(x^5)}$$

$$= 24 + 40\left(\frac{x}{1+x} - \frac{3x^3}{1+x^3} - \frac{7x^7}{1+x^7} + \frac{9x^9}{1+x^9} - \cdots\right)$$

$$\frac{\psi^5(x)}{\psi(x^5)} - 25x^2\psi(x)\psi^3(x^5)$$

$$= 1 + 5\left(\frac{x}{1+x} - \frac{2x^2}{1+x^2} - \frac{3x^3}{1+x^3} + \frac{4x^4}{1+x^4} + \cdots\right)$$

$$\frac{\phi^5(x)}{\phi(x^5)} + 4\cdot\frac{\psi^5(x)}{\psi(x^5)} = 5\frac{\phi^2(x)}{\phi^2(x^5)}$$

$$\phi(x)\phi^3(x^5) + 4x^2\psi(x)\psi^3(x^5)$$

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values of m, n & x,

$$\theta = \frac{n}{(m+1)x+1-n} - \frac{1(1-n)(1+x)}{(m+2)x+3-n} - \frac{2(2-n)(1+x)}{(m+3)x+5-n} - \&c$$

30. $1 + \frac{x}{n+1} + \frac{x^2}{(n+1)(n+2)} + \&c$

$$= \frac{e^x \lfloor n}{x^n} - \cfrac{n}{x+\cfrac{1-n}{1+\cfrac{1}{x+\cfrac{2-n}{1+\cfrac{2}{x+\cfrac{3-n}{1+}}}}}} \&c$$

$$= \frac{e^x \lfloor n}{x^n} - \cfrac{n}{x+1-n-\cfrac{1(1-n)}{x+3-n-\cfrac{2(2-n)}{x+5-n-\cfrac{3(3-n)}{x+7-n-}}}} \&c$$

Cor. $\frac{1}{n} - \frac{x}{\lfloor 2} \cdot \frac{1}{n+1} + \frac{x^2}{\lfloor 2} \cdot \frac{1}{n+2} - \frac{x^3}{\lfloor 3} \cdot \frac{1}{n+3} + \&c$

$$= \frac{\lfloor n-1}{x^n} - \cfrac{e^{-x}}{x+\cfrac{1-n}{1+\cfrac{1}{x+\cfrac{2-n}{1+}}}} \&c = \frac{\lfloor n-1}{x^n} - \cfrac{e^{-x}}{x+1-n-\cfrac{1(1-n)}{x+3-n-}} \&c.$$

Ex. 1. $x - \frac{x^3}{3\lfloor 1} + \frac{x^5}{5\lfloor 2} - \&c = \frac{1}{2}\sqrt{\pi}$ when $x = \infty$

2. $x - \frac{x^3}{3^2 \lfloor 1} + \frac{x^5}{5^2 \lfloor 2} - \&c = \sqrt{\frac{\pi}{4}}\left(\frac{C_0}{2} + \log_e 2x\right)$ when $x = 0$

31. $1 + \frac{x}{1.3} + \frac{x^2}{1.3.5} + \frac{x^3}{1.3.5.7} + \frac{x^4}{1.3.5.7.9} + \&c$

$$= \sqrt{\frac{\pi}{2x}}\, e^x - \cfrac{1}{x+\cfrac{1}{1+\cfrac{2}{x+\cfrac{3}{1+\cfrac{4}{x+}}}}} \&c$$

$$= \sqrt{\frac{\pi}{2x}}\, e^x - \cfrac{1}{x+1-\cfrac{1.2}{x+5-\cfrac{3.4}{x+9-\cfrac{5.6}{x+13-}}}} \&c$$

Cor. $\frac{1}{1+\frac{1}{1+\frac{2}{1+\frac{3}{1+\frac{4}{1+}}}}} \&c = \sqrt{\frac{\pi e}{2}}$

32. $\frac{x}{1\lfloor 1} - \frac{x^2}{2\lfloor 2} + \frac{x^3}{3\lfloor 3} - \frac{x^4}{4\lfloor 4} + \&c = C_0 + \log_e 2 + \frac{e^{-x}}{x+\cfrac{1}{1+\cfrac{2}{x+\cfrac{3}{1+}}}} \&c$

$\therefore C_0 + \log_e x + e^{-x}\phi(x)$.

$\phi(x) = \cfrac{1}{x+1-\cfrac{1^2}{x+3-\cfrac{2^2}{x+5-}}} \&c$

$\phi(x+1) = x\{\phi(x)\}^2$ very nearly when x is great

$\phi(2) - \{\phi(1)\}^2 = \frac{1}{170}$ nearly.

$\phi(x) = \frac{1}{x+1} + \frac{1}{x^3+5x^2+2x+\theta}$ where $\theta = 0$ when $x = \infty$ and $\theta = \frac{1}{10}$ when $x = 0$.

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$$\frac{\psi(x^7)\psi(x^9) - \psi(-x^7)\psi(-x^9)}{\psi(x)\psi(x^{63}) - \psi(-x)\psi(-x^{63})} = x^6$$

$$\frac{\psi(x^5)\psi(x^{11}) - \psi(-x^5)\psi(-x^{11})}{\psi(x)\psi(x^{55}) + \psi(-x)\psi(-x^{55})} = x^5$$

$$\frac{\psi(x^3)\psi(x^{13}) - \psi(-x^3)\psi(-x^{13})}{\psi(x)\psi(x^{39}) + \psi(-x)\psi(-x^{39})} = x^3$$

$$x\psi^5(x)\psi(x^3) - 9x^2\psi(x)\psi^5(x^3)$$

$$= \frac{x}{1-x^2} - \frac{2^2x^2}{1-x^4} + \frac{4^2x^4}{1-x^8} - \frac{5^2x^5}{1-x^{10}} + \&c$$

$$9\phi(x)\phi^5(x^3) - \phi^5(x)\phi(x^3)$$

$$= 8\left(1 + \frac{x}{1+x} - \frac{2^2x^2}{1-x^2} + \frac{4^2x^4}{1-x^4} - \frac{5^2x^5}{1+x^5} + \frac{7^2x^7}{1+x^7} - \&c\right)$$

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If $n = \frac{1}{14-\theta}$ then $x = 100n - 8.6 + \frac{\cdot 81}{9n+\cdot 14}$ very nearly

If $x > 7$, $x = \frac{\cdot 08\theta^2 + 6.34\theta - 4.45}{14-\theta}$ to 2 places of decimals

and if $x < 7$, $14-\theta = \frac{1000}{11x+75}$ to a place of decimal.

$x = 16.74$ when $\theta = 10$ and $\theta = 5.6$ when $x = 4$.

$$\phi(x) = \frac{1}{x} - \frac{11}{x^2} + \frac{\underline{12}}{x^3} - \ldots \pm \frac{1}{x^n}\frac{\underline{n}}{x+n+1} - \frac{1(1+n)}{x+n+3} - \frac{2(2+n)}{x+n+5} - \&c$$

Cor. $\frac{x}{\underline{1}} + \frac{x^2}{\underline{2}}(1+\frac{1}{2}) + \frac{x^3}{\underline{3}}(1+\frac{1}{2}+\frac{1}{3}) + \&c = e^x(C_0 + \log_e x) + \phi(x)$.

33. $\frac{x}{1^n\underline{1}} - \frac{x^2}{2^n\underline{2}} + \frac{x^3}{3^n\underline{3}} - \frac{x^4}{4^n\underline{4}} + \&c = \phi_n(x) + (-1)^{n-1}\psi_n(x)e^{-x}$.

where $\phi_n(x)$ is the term independent of p in $\frac{x^p\underline{-p}}{p^n}$. and

$$\psi_n(x) - \psi_n'(x) = \frac{\psi_{n-1}(x)}{x}.$$

$$\phi_n(x) = \frac{1}{\underline{n}}\left\{A_0(\log_e x)^n + n A_1(\log_e x)^{n-1} + \frac{n(n-1)}{\underline{2}}(\log_e x)^{n-2} + \ldots + A_n\right\}$$

where $\underline{x} = A_0 - A_1\frac{x}{\underline{1}} + A_2\frac{x^2}{\underline{2}} - A_3\frac{x^3}{\underline{4}} + \&c$

$$A_n = C_0 A_{n-1} + (n-1)S_2 A_{n-2} + (n-1)(n-2)S_3 A_{n-3} + \&c$$

$\underline{x} = 1 - \cdot 5772156649\,x + \cdot 9890560173\,x^2$
$- \cdot 9074790803\,x^3 + \cdot 9817280965\,\frac{x^4}{1+\theta x}$.

$\theta = 1$ very nearly. when $x = 0$ $\theta = 1.00027$

$x = 1$; $\theta = \frac{51}{52}$. $x = 2$; $\theta = \frac{77}{82}$. $x = 6$; $\theta = \frac{5}{68}$. $x = 7$; $\theta =$

$$\psi_n(x) = \cfrac{x}{\left\{x + \cfrac{n}{2 + \cfrac{5n+10}{6x + \cfrac{41n+58}{10 + \&c}}}\right\}^{n+1}}$$

Ex $\left(\frac{x}{1^2\underline{1}} - \frac{x^2}{2^2\underline{2}} + \frac{x^3}{3^2\underline{3}} - \frac{x^4}{4^2\underline{4}} + \frac{x^5}{5^2\underline{5}} - \&c\right)$

$- \frac{1}{2}\left(\frac{x}{1\underline{1}} - \frac{x^2}{2\underline{2}} + \frac{x^3}{3\underline{3}} - \frac{x^4}{4\underline{4}} + \frac{x^5}{5\underline{5}} - \ldots\right)^2$

$= \frac{\pi^2}{12}$ when x becomes infinitely great.

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$$1 - \frac{n}{1!}\left(\frac{1}{2}\right)^2 + \frac{n(n-1)}{2!}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 - \&c.$$

$$= \left(\frac{n-\frac{1}{2}}{\frac{1}{2}\lfloor n}\right)^2 \left\{1 + \frac{1^2}{2\cdot 4}\cdot\frac{n\cdot(n+\frac{1}{2})}{(n-\frac{1}{2})^2} + \right.$$

$$\left. \frac{1^2\cdot 3^2}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{n(n-1)\cdot(n+\frac{1}{2})(n-\frac{1}{2})}{(n-\frac{1}{2})^2(n-\frac{3}{2})^2} + \&c \right\}$$

$$\therefore \frac{\pi}{4}\left\{1 + \left(\frac{1}{2}\right)^2\left(\frac{1+x}{2}\right) + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{1+x}{2}\right)^2 + \&c\right\}^2$$

$$- \frac{\pi}{4}\left\{1 + \left(\frac{1}{2}\right)^2\left(\frac{1-x}{2}\right) + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{1-x}{2}\right)^2 + \&c\right\}^2$$

$$= x + \frac{x^3}{2} + \frac{41x^5}{120} + \frac{21x^7}{80} + \&c.$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{x\sin\theta\, d\theta\, d\phi}{(1-x^2\sin^2\theta)\sqrt{1+x^2\sin^2\theta\sin^2\phi}} = f(x)$$

$$\text{Now } f\left(\frac{2x}{1+x^2}\right) = \frac{5}{1+x^2} f(x)$$

$$\therefore \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{x\sin\theta\, d\theta\, d\phi}{\sqrt{1-x^2\sin^2\theta}\sqrt{1-x^2\sin^2\theta\sin^2\phi}}$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\sin^{-1}x} \frac{d\theta}{\sqrt{1-x^2\sin^2\phi - \sin\theta\cos\phi}}\, d\phi.$$

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CHAPTER XV

1. $\left[1+\left(\frac{1}{2}\right)^2\left\{1-\frac{\phi(-x)}{\phi(x)}\right\}+\left(\frac{1.3}{2.4}\right)^2\left\{1-\frac{\phi(-x)}{\phi(x)}\right\}^2+\&c\right]\frac{1}{\sqrt{\phi(x)}}$
is always an even function of x

2. $1+\left(\frac{1}{2}\right)^2\left(1-\frac{1}{x}\right)+\left(\frac{1.3}{2.4}\right)^2\left(1-\frac{1}{x}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left(1-\frac{1}{x}\right)^3+\&c$
$=\sqrt{x}\left\{1+\left(\frac{1}{2}\right)^2(1-x)+\left(\frac{1.3}{2.4}\right)^2(1-x)^2+\left(\frac{1.3.5}{2.4.6}\right)^2(1-x)^3+\&c\right\}$

3. $1+\left(\frac{1}{2}\right)^2\left(\frac{4x}{1+2x+x^2}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{4x}{1+2x+x^2}\right)^2+\&c$
$=(1+x)\left\{1+\left(\frac{1}{2}\right)^2x^2+\left(\frac{1.3}{2.4}\right)^2x^4+\left(\frac{1.3.5}{2.4.6}\right)^2x^6+\&c\right\}$
cor. $1+\left(\frac{1}{2}\right)^2\left\{\frac{8x(1+x^2)}{(1+x)^4}\right\}+\left(\frac{1.3}{2.4}\right)^2\left\{\frac{8x(1+x^2)}{(1+x)^4}\right\}^2+\&c$
$=(1+x)^2\left\{1+\left(\frac{1}{2}\right)^2x^4+\left(\frac{1.3}{2.4}\right)^2x^8+\left(\frac{1.3.5}{2.4.6}\right)^2x^{12}+\&c\right\}$

4. $1+\left(\frac{1}{2}\right)^2\frac{2x}{1+x}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{2x}{1+x}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left(\frac{2x}{1+x}\right)^3+\&c$
$=\sqrt{1+x}\left(1+\frac{1.3}{4^2}x^2+\frac{1.3.5.7}{4.8^2}x^4+\&c\right)$

5. $1+\left(\frac{1}{2}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c$
$=1+\left(\frac{1}{4}\right)^2x+\left(\frac{1.5}{4.8}\right)^2x^2+\left(\frac{1.5.9}{4.8.12}\right)^2x^3+\&c$

6. $1+\left(\frac{1}{2}\right)^3x+\left(\frac{1.3}{2.4}\right)^3x^2+\left(\frac{1.3.5}{2.4.6}\right)^3x^3+\&c$
$=\left\{1+\left(\frac{1}{2}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c\right\}^2$

7. Let $\pi\alpha\beta=1$ and $\alpha=\frac{\sqrt{\pi}}{\left(\frac{1}{4}\right)^2}$ such that
$\alpha=1.180340,599016,092$
$\beta=.269676,300594,191$
$1/\beta=3.708,149,354602,731$
$1+\left(\frac{1}{2}\right)^2\left(\frac{1+x}{2}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1+x}{2}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left(\frac{1+x}{2}\right)^3+\&c$
$=\alpha\left\{1+\frac{1^2}{2.4}x^2+\frac{1^2.5^2}{2.4.6.8}x^4+\frac{1^2.5^2.9^2}{2.4.6.8.10.12}x^6+\&c\right\}$
$+\beta\left\{x+\frac{3^2}{4.6}x^3+\frac{3^2.7^2}{4.6.8.10}x^5+\frac{3^2.7^2.11^2}{4.6.8.10.12.14}x^7+\&c\right\}$

1·3. There are two solutions for this form &

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$$\psi(x)\psi(x^5) - \psi(-x)\psi(-x^5)$$
$$= 2x f(x^4, x^{20}) + 4x^{15}\psi(x^6)\psi(x^{10})$$

$$\psi(p)\psi(q) = \psi(pq) + pf(q/p, p^2) + p^3 q f\left(\frac{q^2}{p^2}, \frac{p^2}{q}\right)$$
$$+ p^6 q^3 f\left(\frac{q^3}{p^3}, \frac{p^3}{q^2}\right) + \&c.$$

$$\phi(p)\phi(q) = \phi(pq) + 2pf(q/p, p^3 q) +$$
$$2pq f\left(\frac{p^3}{q}, \frac{q^3}{p^3}\right) + 2p^4 q f\left(\frac{q^3}{p^3}, \frac{p^5}{q}\right) +$$
$$2p^4 q^4 f\left(\frac{p^5}{q^3}, \frac{q^5}{p^3}\right) + 2p^9 q^4 f\left(\frac{q^5}{p^{10}}, \frac{p^7}{q^3}\right)$$
$$+ 2p^9 q^9 f\left(\frac{p^7}{q^{10}}, \frac{q^7}{p^7}\right) + \&c$$

$$\sqrt[4]{1+x^2}\left\{1 + \left(\frac{1}{4}\right)^2 \frac{1 \pm i}{2} x^2 \right\}$$
$$= \frac{1+i}{2}\left\{1 + \left(\frac{1}{4}\right)^2 \frac{1 + \frac{x}{\sqrt{1+x^2}}}{2} + \&c\right\}$$
$$+ \frac{1-i}{2}\left\{1 + \left(\frac{1}{4}\right)^2 \frac{1 - \frac{x}{\sqrt{1+x^2}}}{2} + \&c\right\}$$

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8. $1+\left(\frac{1}{2}\right)^2\left\{\frac{(1+x)^2}{2(1+x^4)}\right\}+\left(\frac{1.3}{2.4}\right)^2\left\{\frac{(1+x)^2}{2(1+x^4)}\right\}^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left\{\frac{(1+x)^2}{2(1+x^4)}\right\}^3+\&c$

$=\alpha\sqrt{1+x^2}\left(1+\frac{1}{2}\cdot\frac{1}{3}x^4+\frac{1.3}{2.4}\cdot\frac{1.5}{3.7}x^8+\frac{1.3.5}{2.4.6}\cdot\frac{1.5.9}{3.7.11}x^{12}+\&c\right)$

$+2\beta x\sqrt{1+x^2}\left(1+\frac{1}{2}\cdot\frac{3}{5}x^4+\frac{1.3}{2.4}\cdot\frac{3.7}{5.9}x^8+\frac{1.3.5}{2.4.6}\cdot\frac{3.7.11}{5.9.13}x^{12}+\&c\right)$

Ex. 1. $1+\left(\frac{1}{2}\right)^2\frac{x}{1+x}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{x}{1+x}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left(\frac{x}{1+x}\right)^3+\&c$

$=\sqrt{\frac{1+x}{1-x}}\left\{1-\frac{1.3}{4^2}\cdot\left(\frac{4x}{1-2x+x^2}\right)+\frac{1.3.5.7}{4^2.8^2}\cdot\left(\frac{4x}{1-2x+x^2}\right)^2-\&c\right\}$

2. $1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\left(\frac{1.3.5}{2.4.6}\right)^2x^3+\&c$

$=\frac{1}{\sqrt{1+x}}\left[1+\frac{1.3}{4^2}\cdot\left\{\frac{4x}{(1+x)^2}\right\}+\frac{1.3.5.7}{4^2.8^2}\left\{\frac{4x}{(1+x)^2}\right\}^2+\&c\right]$

3. $1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\left(\frac{1.3.5}{2.4.6}\right)^2x^3+\&c$

$=\frac{1}{\sqrt{1-x}}\left[1-\left(\frac{1}{4}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}+\left(\frac{11.5}{6.8}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}^2-\&c\right]$

4. $1+\left(\frac{1}{2}\right)^2\frac{1-\sqrt{1-x}}{2}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c$

$=\sqrt{1-x}\left\{1+\left(\frac{3}{4}\right)^2x+\left(\frac{3.7}{4.8}\right)^2x^2+\left(\frac{3.7.11}{4.8.12}\right)^2x^3+\&c\right\}$

5. $1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\left(\frac{1.3.5}{2.4.6}\right)^2x^3+\&c$

$=\frac{(1+x)}{(1-x)\sqrt{1-x}}\left[1-\left(\frac{3}{4}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}+\left(\frac{3.7}{4.8}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}^2-\&c\right]$

6. $1+\left(\frac{1}{2}\right)^2\frac{1+x}{2}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1+x}{2}\right)^2+\&c$

$=\frac{\alpha}{\sqrt{1-x^2}}\left\{1-\frac{1^2}{2.4}\cdot\frac{x^2}{1-x^2}+\frac{1^2.5^2}{2.4.6.8}\left(\frac{x^2}{1-x^2}\right)^2-\&c\right\}$

$+\frac{\beta}{\sqrt{1-x^2}}\cdot\frac{x}{\sqrt{1-x^2}}\left\{1-\frac{3^2}{4.6}\cdot\frac{x^2}{1-x^2}+\frac{3^2.7^2}{4.6.8.10}\left(\frac{x^2}{1-x^2}\right)^2-\&c\right\}$

7. $1+\left(\frac{1}{2}\right)^2\left\{\frac{(1+x)^2}{2(1+x^4)}\right\}+\left(\frac{1.3}{2.4}\right)^2\left\{\frac{(1+x)^2}{2(1+x^4)}\right\}^2+\&c$

$=\frac{\alpha}{\sqrt{1-x^2}}\left\{1-\frac{1}{2}\cdot\frac{2}{3}\left(\frac{x^4}{1-x^4}\right)+\frac{1.3}{2.4}\cdot\frac{1.6}{3.7}\left(\frac{x^4}{1-x^4}\right)^2-\&c\right\}$

$+\frac{2\beta x}{\sqrt{1-x^2}}\left\{1-\frac{1}{4}\cdot\frac{2}{5}\left(\frac{x^4}{1-x^4}\right)+\frac{1.3}{2.4}\cdot\frac{2.6}{5.9}\left(\frac{x^4}{1-x^4}\right)^2-\&c\right\}$

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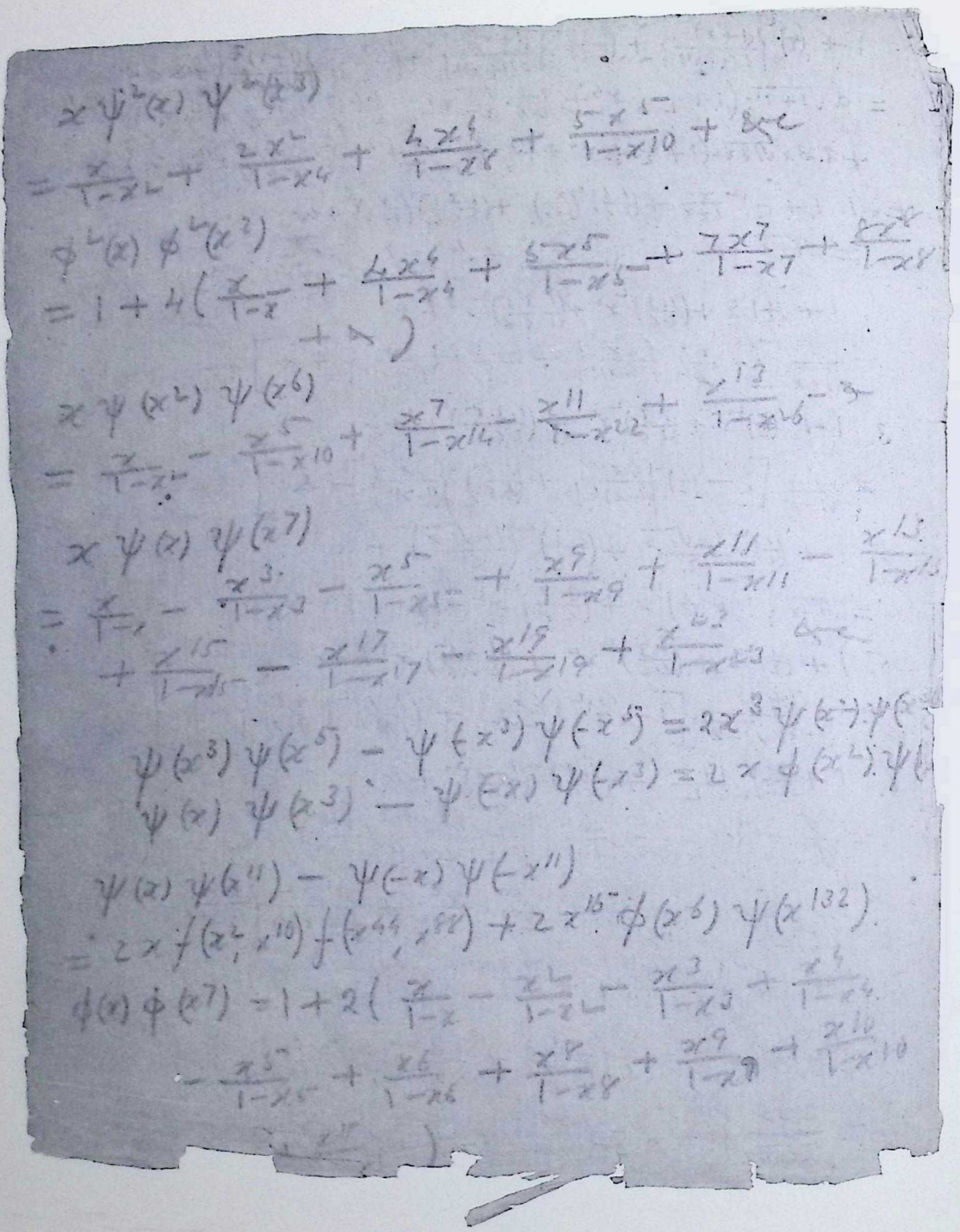

$$x\psi^2(x)\psi^2(x^3)$$
$$= \frac{x}{1-x^2} + \frac{2x^2}{1-x^4} + \frac{4x^4}{1-x^6} + \frac{5x^5}{1-x^{10}} + \&c$$

$$\phi^2(x)\phi^2(x^3)$$
$$= 1 + 4\left(\frac{x}{1-x} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \frac{7x^7}{1-x^7} + \frac{8x^8}{1-x^8} + \&c\right)$$

$$x\psi(x^2)\psi(x^6)$$
$$= \frac{x}{1-x^2} - \frac{x^5}{1-x^{10}} + \frac{x^7}{1-x^{14}} - \frac{x^{11}}{1-x^{22}} + \frac{x^{13}}{1-x^{26}} - \&c$$

$$x\psi(x)\psi(x^7)$$
$$= \frac{x}{1-x} - \frac{x^3}{1-x^3} - \frac{x^5}{1-x^5} + \frac{x^9}{1-x^9} + \frac{x^{11}}{1-x^{11}} - \frac{x^{15}}{1-x^{15}}$$
$$+ \frac{x^{15}}{1-x^{15}} - \frac{x^{17}}{1-x^{17}} - \frac{x^{19}}{1-x^{19}} + \frac{x^{21}}{1-x^{21}} \&c$$

$$\psi(x^3)\psi(x^5) - \psi(-x^3)\psi(-x^5) = 2x^3\psi(x^7)\psi(x^{15})$$
$$\psi(x)\psi(x^3) - \psi(-x)\psi(-x^3) = 2x\phi(x^4)\psi(x^6)$$

$$\psi(x)\psi(x^{11}) - \psi(-x)\psi(-x^{11})$$
$$= 2x f(x^2, x^{10}) f(x^{44}, x^{88}) + 2x^{15}\phi(x^{66})\psi(x^{132})$$

$$\phi(x)\phi(x^7) = 1 + 2\left(\frac{x}{1-x} - \frac{x^2}{1-x^2} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^4}\right.$$
$$\left. - \frac{x^5}{1-x^5} + \frac{x^6}{1-x^6} + \frac{x^8}{1-x^8} + \frac{x^9}{1-x^9} + \frac{x^{10}}{1-x^{10}}\right.$$
$$\left. + \frac{x^{15}}{1-x^{15}}\right)$$

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$= 2(2-\sqrt{2})\{\ldots$

9. $1+\left(\frac{1}{2}\right)^2\frac{80}{81}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{80}{81}\right)^2+\&c=\frac{18}{\sqrt{62}}\left\{1+\frac{1.3}{4^2}\cdot\frac{1}{31^2}+\frac{1.3.5.7}{4^2.8^2}\cdot\frac{1}{31^4}+\ldots\right.$

10. $1+\left(\frac{1}{2}\right)^2\left(\frac{15}{16}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{15}{16}\right)^2+\&c=\frac{32}{\sqrt{322}}\left\{1+\frac{1.3}{4^2}\cdot\frac{1}{161^2}+\frac{1.3.5.7}{4^2.8^2}\cdot\frac{1}{161^4}\right.$

11. $1+\left(\frac{1}{2}\right)^2\frac{1}{3}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{5}+\left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{7}+\&c$

$$=\frac{1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\&c}{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\&c}$$

12. $1+\left(\frac{1}{2}\right)^2\frac{1}{5}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{9}+\left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{13}+\&c=\frac{\lfloor -\frac{3}{4}}{(\lfloor -\frac{1}{4})^3\sqrt{2}}=\alpha^2$

13. $1-\left(\frac{1}{2}\right)^3+\left(\frac{1.3}{2.4}\right)^3-\left(\frac{1.3.5}{2.4.6}\right)^3+\&c=\left(\frac{\lfloor\frac{1}{8}}{\lfloor\frac{1}{4}\lfloor -\frac{1}{8}}\right)^2$

14. $1-5\left(\frac{1}{2}\right)^3+9\left(\frac{1.3}{2.4}\right)^3-13\left(\frac{1.3.5}{2.4.6}\right)^3+\&c=\frac{2}{\pi}$

15. $1+\left(\frac{1}{4}\right)^2+\left(\frac{1.5}{4.8}\right)^2+\left(\frac{1.5.9}{4.8.12}\right)^2+\&c=\frac{\sqrt{\pi}}{(\lfloor -\frac{1}{4})^2}=\alpha$

16. $1-\left(\frac{1}{4}\right)^2+\left(\frac{1.5}{4.8}\right)^2-\left(\frac{1.5.9}{4.8.12}\right)^2+\&c=\frac{\lfloor\frac{1}{8}}{\lfloor\frac{1}{4}\lfloor -\frac{1}{8}}$

17. $1+\left(\frac{1}{4}\right)^3+\left(\frac{1.5}{4.8}\right)^3+\left(\frac{1.5.9}{4.8.12}\right)^3+\&c=\frac{\sqrt{\pi}}{\lfloor -\frac{1}{4}(\lfloor -\frac{1}{8})^2\sqrt[4]{2}}$

18. $1-9\cdot\left(\frac{1}{4}\right)^3+17\left(\frac{1.5}{4.8}\right)^3-25\left(\frac{1.5.9}{4.8.12}\right)^3+\&c=\frac{2\sqrt{2}}{\sqrt{\pi}}$

19. $1+9\cdot\left(\frac{1}{4}\right)^4+17\left(\frac{1.5}{4.8}\right)^4+25\left(\frac{1.5.9}{4.8.12}\right)^4+\&c=\frac{4}{\sqrt{2\pi}(\lfloor -\frac{1}{4})^2}$

20. $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{d\theta\, d\phi}{\sqrt{1-\sin^2\theta\sin^2\phi\sin^2\psi}}=\left\{\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\sin^2\theta\sin^2\frac{\psi}{2}}}\right.$

21. $\int_0^{\frac{\pi}{4}}\left\{\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\sin^2\theta\sin^2\phi}}\right\}^2 d\phi=\frac{\pi^3}{16}\left\{1+\left(\frac{1}{2}\right)^4+\left(\frac{1.3}{2.4}\right)^4+\&c\right.$

9. If $\phi(x)$ can be expressed in n different ways the value in the nth way being c_n+v_n and if $c_1, c_2, \ldots$ to be similar and $v_1, v_2, v_3 \ldots v_n$ are known to be then $c_1, c_2, c_3 \ldots c_n$ must be identically equal (...) and the real value of $\phi(x) = c + v_1 + v_2 + v_3 + \ldots$

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$$+ \phi(1)\frac{\lfloor a+n \rfloor b+n}{\lfloor n-1} - \frac{\phi(a+1)}{\lfloor 1} \frac{\lfloor a+n+1 \rfloor b+n+1}{\lfloor n-2} + \&c$$

$$\frac{\lfloor a+n \rfloor b+n}{} \left\{ \phi(0) \frac{\lfloor a \rfloor b}{\lfloor a+b+n+1} + \frac{\phi(0)-\phi(1)}{\lfloor 1} \frac{\lfloor a+1 \rfloor b+1}{\lfloor a+b+n+2} \right.$$

$$\left. + \frac{\phi(0)-2\phi(1)+\phi(2)}{\lfloor 2} \cdot \frac{\lfloor a+2 \rfloor b+2}{\lfloor a+b+n+2} + \&c \right\}$$

The irreducible part in $1-2y\left\{\frac{y}{(1-y)^2}\cos 2\theta + \frac{y^2}{(1-y^2)^2}\cos 4\theta + \&c\right\} = \frac{\theta^2}{6}\left\{1-24\left(\frac{y}{1-y}+\frac{2y^2}{1-y^2}+\&c\right)\right\}$ 2

$$+\theta\left\{1-24\left(\frac{y}{1-y}+\frac{2y^2}{1-y^2}+\&c\right)\right\}\left\{\cos\theta + 4\left(\frac{y}{1-y}\sin 2\theta + \frac{y^2}{1-y^2}\sin 4\theta + \&c\right)\right\}$$

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and $Q_r = \phi(r) + \ldots \phi(r+1) + \ldots$

$\phi(0) + (1-x)\phi(1) + (1-x)^2\phi(2) + (1-x)^3\phi(3) + \&c$

$= Q_0 - Q_1 x + Q_2 x^2 - Q_3 x^3 + \&c$

$+ \frac{1}{\left(\log_e \frac{1}{1-x}\right)^{n+1}} \left\{ P_0 + P_1 \log_e \frac{1}{1-x} + P_2 \left(\log_e \frac{1}{1-x}\right)^2 + P_3 \left(\log_e \frac{1}{1-x}\right)^3 + \&c \right\}$

Cor. 1. If $Q'_r = \frac{\lfloor m}{\lfloor m-r-1 \lfloor r}\phi(m) + \frac{\lfloor m+1}{\lfloor m-r+1 \lfloor r}\phi(m+1) + \frac{\lfloor m+2}{\lfloor m-r+2 \lfloor r}\phi(m+2) + \&c$

then $\phi(m)(1-x)^m + \phi(m+1)(1-x)^{m+1} + \phi(m+2)(1-x)^{m+2} + \&c$

$= Q_0 - Q_1 x + Q_2 x^2 - Q_3 x^3 + \&c$

$+ \frac{1}{\left(\log_e \frac{1}{1-x}\right)^{n+1}} \left\{ P_0 + P_1 \log_e \frac{1}{1-x} + P_3 \left(\log_e \frac{1}{1-x}\right)^2 + \&c \right\}$

Cor. 2. If $a+b+c+1 = d+e$, then

$\left\{ \frac{\lfloor a \lfloor b \lfloor c}{\lfloor d \lfloor e} + \frac{1-x}{\lfloor 1} \cdot \frac{\lfloor a+1 \lfloor b+1 \lfloor c+1}{\lfloor d+1 \lfloor e+1} + \frac{(1-x)^2}{\lfloor 2} \frac{\lfloor a+2 \lfloor b+2 \lfloor c+2}{\lfloor d+2 \lfloor e+2} + \&c \right\}$

$+ \log_e x$ when $x = 0$

$= -\Sigma \frac{1}{a} - \Sigma \frac{1}{b} + 1 \cdot \frac{(c-d)(c-e)}{(a+1)(b+1)} + \frac{1}{2} \cdot \frac{(c-d)(c-d-1)}{(a+1)(a+2)} \cdot \frac{(c-e)}{(b+1)} + \&c$

11. $\lfloor a \lfloor b \left\{ \frac{\lfloor a+n \lfloor b+n}{\lfloor a+b+n+1} + \frac{1-x}{\lfloor 1} \frac{\lfloor a+n+1 \lfloor b+n+1}{\lfloor a+b+n+2} + \&c \right\}$.

$= \left\{ \lfloor a+n \lfloor b+n \lfloor -n-1 - \frac{x}{\lfloor 1} \lfloor a+n+1 \lfloor b+n+1 \lfloor -n-2 + \&c \right\}$

$+ \frac{1}{x^n} \left\{ \lfloor a \lfloor b \lfloor n-1 - \frac{x}{\lfloor 1} \lfloor a+1 \lfloor b+1 \lfloor n-2 + \&c \right\}$.

This is true for all values of n.

N. B. If n is an integer R.H.S takes the form $\infty - \infty$ should write $n+h$ for n first of all and then simplification h should be made to vanish

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Cor. 1. If n is a positive integer,

$$\lfloor a \lfloor b \left\{ \frac{\lfloor a+n \lfloor b+n}{\lfloor a+b+n+1} + \frac{1-x}{11} \cdot \frac{\lfloor a+n+1 \lfloor b+n+1}{\lfloor a+b+n+2} + \&c \right\}$$

$$+ (-1)^n (\log_e x) \left\{ \frac{\lfloor a+n \lfloor b+n}{\lfloor n} + \frac{x}{11} \cdot \frac{\lfloor a+n+1 \lfloor b+n+1}{\lfloor n+1} + \&c \right\}$$

$$+ (-1)^n \left\{ \frac{\lfloor a+n \lfloor b+n}{\lfloor n} \left(\Sigma \frac{1}{a+n} + \Sigma \frac{1}{b+n} - \Sigma \frac{1}{n} - 0 \right) \right.$$

$$+ \frac{x}{11} \cdot \frac{\lfloor a+n+1 \lfloor b+n+1}{\lfloor n+1} \left(\Sigma \frac{1}{a+n+1} + \Sigma \frac{1}{b+n+1} - \Sigma \frac{1}{n+1} - \Sigma \frac{1}{1} \right)$$

$$+ \frac{x^2}{12} \cdot \frac{\lfloor a+n+2 \lfloor b+n+2}{\lfloor n+2} \left(\Sigma \frac{1}{a+n+2} + \Sigma \frac{1}{b+n+2} - \Sigma \frac{1}{n+2} - \Sigma \frac{1}{2} \right)$$

$$\left. + \&c \right\}$$

$$= \frac{1}{x^n} \left\{ \lfloor a \lfloor b \lfloor n-1 - \frac{x}{11} \lfloor a+1 \lfloor b+1 \lfloor n-2 + \&c \text{ to } n \text{ terms} \right\}$$

Cor 2. If n is a negative integer,

$$\lfloor a \lfloor b \left\{ \frac{\lfloor a+n \lfloor b+n}{\lfloor a+b+n+1} + \frac{1-x}{11} \cdot \frac{\lfloor a+n+1 \lfloor b+n+1}{\lfloor a+b+n+2} + \&c \right\}$$

$$+ (-x)^{-n} (\log_e x) \left\{ \frac{\lfloor a \lfloor b}{\lfloor -n} + \frac{x}{11} \cdot \frac{\lfloor a+1 \lfloor b+1}{\lfloor -n+1} + \&c \right\}$$

$$+ (-x)^{-n} \left\{ \frac{\lfloor a \lfloor b}{\lfloor -n} \left(\Sigma \frac{1}{a} + \Sigma \frac{1}{b} - \Sigma \frac{1}{-n} - 0 \right) \right.$$

$$+ \frac{x}{11} \cdot \frac{\lfloor a+1 \lfloor b+1}{\lfloor -n+1} \left(\Sigma \frac{1}{a+1} + \Sigma \frac{1}{b+1} - \Sigma \frac{1}{-n+1} - \Sigma \frac{1}{1} \right)$$

$$+ \frac{x^2}{12} \cdot \frac{\lfloor a+2 \lfloor b+2}{\lfloor -n+2} \left(\Sigma \frac{1}{a+2} + \Sigma \frac{1}{b+2} - \Sigma \frac{1}{-n+2} - \Sigma \frac{1}{2} \right)$$

$$\left. + \&c \right\}$$

$$= \lfloor a+n \lfloor b+n \lfloor -n-1 - \frac{x}{11} \cdot \lfloor a+n+1 \lfloor b+n+1 \lfloor -n-2 + \&c \text{ to } -n \text{ terms}$$

If n is a fractional [illegible]

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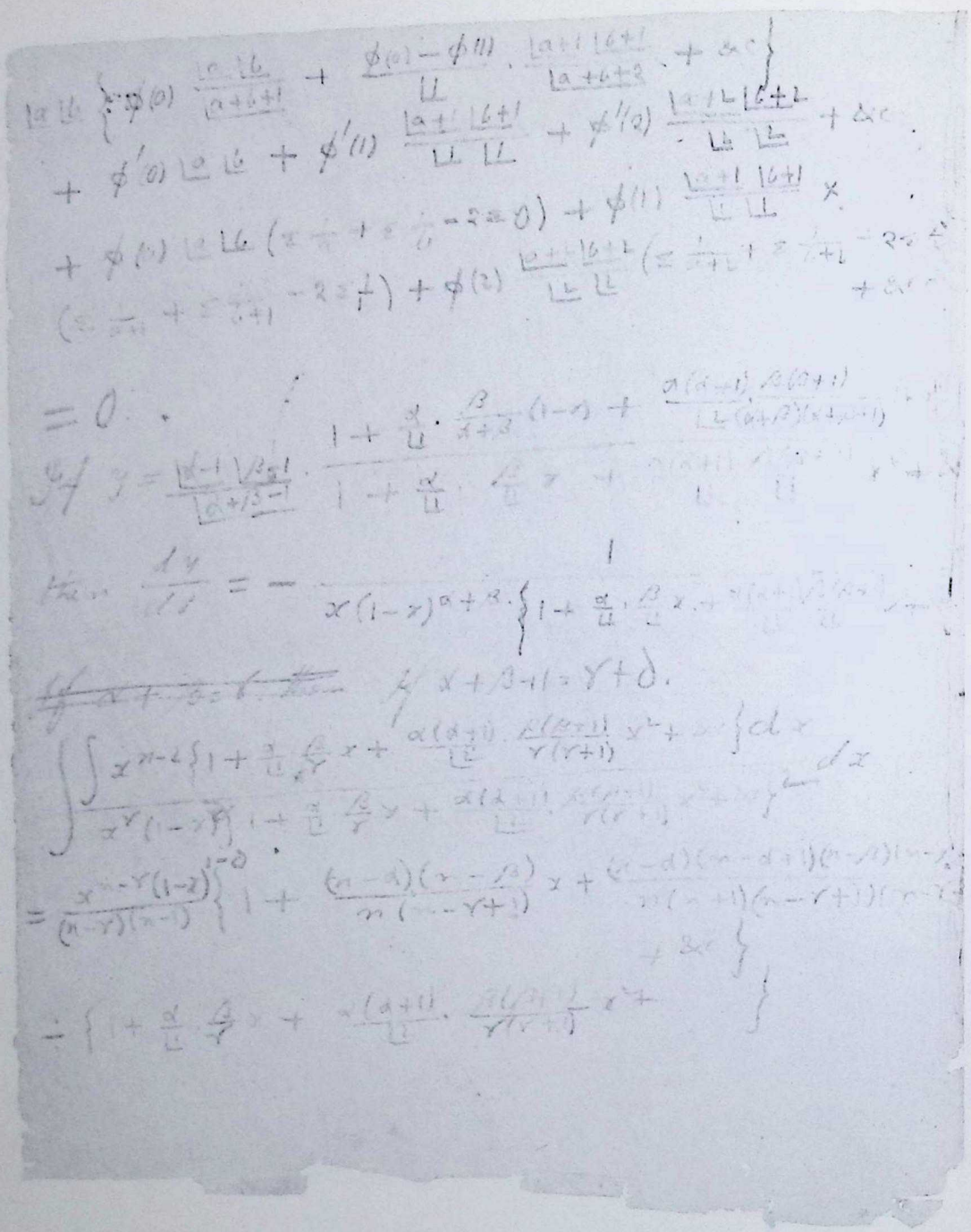

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12. $\lfloor a \lfloor b \left\{ \frac{\lfloor a \lfloor b}{\lfloor a+b+1} + \frac{1-x}{\lfloor 1} \cdot \frac{\lfloor a+1 \lfloor b+1}{\lfloor a+b+2} + \frac{(1-x)^2}{\lfloor 2} \frac{\lfloor a+2 \lfloor b+2}{\lfloor a+b+3} + \&c \right\}$

$+ (\log_e x) \left\{ \lfloor a \lfloor b + x \frac{\lfloor a+1 \lfloor b+1}{\lfloor 1 \lfloor 1} + x^2 \frac{\lfloor a+2 \lfloor b+2}{\lfloor 2 \lfloor 2} + \&c \right\}$

$+ \lfloor a \lfloor b \left(\Sigma \frac{1}{a} + \Sigma \frac{1}{b} - \lfloor 0 \right) + x \frac{\lfloor a+1 \lfloor b+1}{\lfloor 1 \lfloor 1} \left(\Sigma \frac{1}{a+1} + \Sigma \frac{1}{b+1} - \lfloor \Sigma 1 \right)$

$+ x^2 \frac{\lfloor a+2 \lfloor b+2}{\lfloor 2 \lfloor 2} \left(\Sigma \frac{1}{a+2} + \Sigma \frac{1}{b+2} - \lfloor \Sigma \frac{1}{2} \right) + \&c$

$= 0.$

Cor. $\pi \left\{ 1 + \left(\frac{1}{2}\right)^2 (1-x) + \left(\frac{1.3}{2.4}\right)^2 (1-x)^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 (1-x)^3 + \&c \right\}$

$= \left(\log_e \frac{16}{x}\right) \left\{ 1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 x^3 + \&c \right\}$

$- 4 \left\{ \left(\frac{1}{2}\right)^2 \frac{1}{1.2} x + \left(\frac{1.3}{2.4}\right)^2 \left(\frac{1}{1.2} + \frac{1}{3.4}\right) x^2 + \&c \right\}.$

Ex. 1. $4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\tan \frac{\phi}{2}}{\sqrt{1 - k \cos^2\theta \cos^2\phi}} \, d\theta \, d\phi$

$= \pi \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-(1-k)\sin^2\theta}} + \log_e k \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k\sin^2\theta}}$

2. $\left\{ \frac{\lfloor a \lfloor b}{\lfloor a+b+1} + \frac{\lfloor a+1 \lfloor b+1}{\lfloor 1 \lfloor a+b+2} + \frac{\lfloor a+2 \lfloor b+2}{\lfloor 2 \lfloor a+b+3} + \&c \text{ to } n \text{ terms} \right\}$

$- \left\{ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c \text{ to } n \text{ terms} \right\}$ when $n = \infty$

$= - \Sigma \frac{1}{a} - \Sigma \frac{1}{b}.$

3. $\pi \left\{ 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 + \&c \right\} - \left(1 + \frac{1}{2} + \frac{1}{3} + \&c\right) = 4$

13. If $\phi(n) = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 + \&c$ to n terms, then

i. $\pi \phi\left(\frac{n}{4} + \frac{1}{4}\right) = 3 \log_e 2 + \Sigma \frac{1}{\frac{n-1}{2}} + \frac{3}{4n^2} - \frac{99}{32n^4} + \frac{999}{32n^6} - \frac{61224}{[illegible]}$

$= 5 \log_e 2 + 2 \Sigma \frac{1}{\frac{n-2}{4}} - \Sigma \frac{1}{\frac{n-1}{2}} - \frac{2}{8n^2+1^2}$

ii. $\phi(n) = \left(\frac{\lfloor n-\frac{1}{2}}{\lfloor n-1}\right)^2 \left\{ \frac{1}{n} + \left(\frac{1}{2}\right)^2 \frac{1}{n+1} + \left(\frac{1.3}{2.4}\right)^2 \ldots \right.$

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iii. $\phi\left(\frac{1}{4}\right) = \frac{1}{2}$

iv. $\phi\left(\frac{1}{2}\right) = \dfrac{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c}{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c}$

v. $\phi(n+1) = \phi(n) + \frac{1}{n}\left(\frac{\lfloor n-\frac{1}{2}}{\lfloor n}\right)^2$

vi. $1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2.4}{3.5}\right)^2 + \left(\frac{2.4.6}{3.5.7}\right)^2 + \&c$ to n terms $= \frac{\pi^2}{4}\phi(n+\frac{1}{2})$

$- 2\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c\right)$

vii. $1 + (4,3)^2\left\{1 + \left(\frac{3}{5}\right)^2 + \left(\frac{3.7}{5.9}\right)^2 + \left(\frac{3.7.11}{5.9.13}\right)^2 + \&c \text{ to } n \text{ terms}\right\} = 2\phi(n+1)$

14. $\dfrac{a}{1-m} + \dfrac{(a+b)x}{1-mx} + \dfrac{(a+2b)mx^2}{1-mx^2} + \dfrac{(a+3b)m^3}{1-mx^3} + \&c$

$= a \cdot \dfrac{1-mx}{(1-m)(1-x)} + (a+b)\dfrac{1-mx^2}{(1-mx)(1-x)}(mx) + (a+2b)\dfrac{1-mx^3}{(1-mx^2)(1-x^2)}$

$\times (mx^2)^2 + (a+3b)\dfrac{1-mx^4}{(1-mx^3)(1-x^3)}(mx^3)^3 + \&c$

$+ \frac{b}{m}\left\{\dfrac{mx}{(1-x)^2} + \dfrac{(mx)^2}{(1-x^2)^2} + \dfrac{(mx^2)^3}{(1-x^4)^2} + \dfrac{(mx^3)^4}{(1-x^3)^2} + \&c\right\}$

Cor 1. $\dfrac{a}{1-m} + \dfrac{(a+b)x}{1-mx} + \dfrac{(a+2b)mx^2}{1-mx^2} + \&c$

$= a \cdot \dfrac{1+x}{1-m} + (a+b)\dfrac{1+mx}{1-mx}(mx^2) + (a+2b)\dfrac{1+mx^2}{1-mx^2}(mx^4)^2 + \&c$

$+ b\left\{\dfrac{x}{(1-x)^2} + \dfrac{m^2x^2}{(1-mx)^2} + \dfrac{m^5x^6}{(1-mx^2)^2} + \dfrac{m^7x^{12}}{(1-mx^3)^2} + \&c\right\}$

Cor 2. If A_n denotes the no. of factors in n including 1 and n

then $\dfrac{A_1}{x} + \dfrac{A_2}{x^2} + \dfrac{A_3}{x^3} + \dfrac{A_4}{x^4} + \&c$

$= \dfrac{1}{x-1} + \dfrac{1}{x^2-1} + \dfrac{x^3}{x^3-1} + \dfrac{x^4}{x^4-1} + \&c$

hence show that $\dfrac{1}{x-1} + \dfrac{x^2}{x^2-1} + \dfrac{x^3}{x^3-1} + \&c$

$= \dfrac{1}{x} \cdot \dfrac{x+1}{x-1} + \dfrac{1}{x^2}\cdot\dfrac{x^2+1}{x^2-1} + \dfrac{1}{x^3}\cdot\dfrac{x^3+1}{x^3-1} + \&c$

Cor 3. $\dfrac{1}{e^{2x}-1} + \dfrac{1}{e^{4x}-1} + \dfrac{1}{e^{6x}-1} + \&c = e^{-x}\coth x + e^{-2x}\coth 2x + e^{-3x}\coth 3x + \&c$

$= \dfrac{1+b}{1-a}\cdot\dfrac{1+bx}{1-ax}\cdot\dfrac{1+bx^2}{1-ax^2}\cdot\dfrac{1+bx^3}{1-ax^3}\cdot\&c$

$= 1 + \dfrac{a+b}{1-x} + \dfrac{a+b}{1-x}\cdot\dfrac{a+bx}{1-x^2} + \dfrac{a+b}{1-x}\cdot\dfrac{a+bx}{1-x^2}\cdot\dfrac{a+bx^2}{1-x^3} + \&c$

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$$f(x^3, x^6) = \psi(x) - x\psi(x^9)$$

$$(a+1)(b+1)(c+1) + (a-1)(b-1)(c-1)$$
$$= 2(a+b+c+abc)$$

$$\frac{\phi^2(x)}{\phi^2(y)} + \frac{\phi^2(y)}{\phi^2(z)} + \frac{\phi^2(z)}{\phi^2(x)} + \frac{\phi^2(x)\phi^2(y)\phi^2(z)}{\phi^2(x)\phi^2(y)\phi^2(z)}$$

$$= 4\,\frac{\phi^2(x)\phi^2(y)\phi^2(z)}{\phi^2(x)\phi^2(y)\phi^2(z)} + 256xyz\,\frac{\psi^2(xy)\psi^2(yz)\psi^2(zx)}{\phi^2(x)\phi^2(y)\phi^2(z)}$$

$$f(x^2, x^3)\, f(x, x^4) = f(x, x^2)\, f(x^5, x^{14})$$

$$f(x, x^4)\, f(x^2, x^5)\, f(x^3, x^7) = f(x, x^4)\, f^2(x^7, x^{16})$$

$$f(x, x^5) = \psi(x^3)\sqrt[3]{\frac{\phi(x)}{\psi(x)}}$$

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16. Let $f(p,q) = 1 + (p+q) + pq(p^2+q^2) + (pq)^3(p^3+q^3) + (pq)^6(p^4+q^4)$
$+ (pq)^{10}(p^5+q^5) + \&c$, then

i. $f(p,q) = f(q,p)$. ii. $f(1,p) = f(p,p^3)$. iii. $f(-1,p) = 0$.

iv. If n is any integer, then
$f(p,q) = p^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}} f\left\{p(pq)^n, q(pq)^{-n}\right\}$.

These 4 results are evident from the series itself.

v. $f(p,q) = (1+p)(1+kp)(1+k^2p)(1+k^3p)$ &c.
$\times (1+q)(1+kq)(1+k^2q)(1+k^3q)$ &c
$\times (1-pq)(1-kpq)(1-k^2pq)(1-k^3pq)$ &c.
where $k = pq$.

Sol. Since $f(-1,p) = 0$ by iii, we see from iv that if $p(pq)^n = -1$, $q(pq)^n = -1$, then $f(p,q) = 0$ $\therefore (1+pk^n)$ & $(1+qk^n)$ are factors of $f(p,q)$. Again we see that if $(pq)^n = 1$, then
$f(p,q)\left\{1 - \left(\frac{p}{q}\right)^{\frac{n}{2}}\right\} = 0$ $\therefore f(p,q) = 0$. $\therefore 1 - \left(\frac{p}{q}\right)^{\frac{n}{2}}$ is a factor. $f(p,q) = \prod_{n=1}^{n=\infty}(1+pk^n)(1+qk^n)(1-pqk^n)$

vi. $f(p,k^{n-1}q) \times f(kp, k^{n-2}q) \times f(k^2p, k^{n-3}q) \times \ldots\ldots f(k^{n-1}p, q)$
$= f(p,q)\,\frac{\{f(-k^n, -k^{2n})\}^n}{f(-k,-k^2)}$

vii. $f(p,q) - f\left(p^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}}, p^{\frac{n(n-1)}{2}} q^{\frac{n(n+1)}{2}}\right)$
$= pf\left\{\frac{q}{p}, p^{\frac{n(n+3)}{2}} q^{\frac{n(n+1)}{2}}\right\} + p^3 q f\left\{\frac{q^2}{p^2}, p^{\frac{n(n+1)}{2}} q^{\frac{n(n+3)}{2}}\right\}$
$+ p^6 q^3 f\left\{\frac{q^3}{p^3}, p^{\frac{n(n+7)}{2}} q^{\frac{n(n+1)}{2}}\right\}$ + &c to $(n-1)$ terms.

viii. If $pq = rs$, $f(p,q)f(r,s) + f(-p,-q)f(-r,-s)$
$= 2f(pr, qs)f(qr, ps)$

ix. If $pq = rs$, $f(p,q)f(r,s) - f(-p,-q)f(-r,-s)$
$= 2p\, f\left\{\frac{q}{r}, \frac{s}{q}(p,rs)\right\} f\left\{\frac{r}{q}, \frac{q}{s}(p,rs)\right\}$

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$$e^{-\frac{\pi}{\sqrt{2}}} \cdot \frac{1+\frac{1\cdot3}{4\cdot2}(1-x^2)+\frac{1\cdot3\cdot5\cdot7}{4\cdot8\cdot2}(1-x^2)^2+\&c}{1+\frac{1\cdot3}{4\cdot2}x^2+\frac{1\cdot3\cdot5\cdot7}{4\cdot4\cdot2}x^4+\&c}$$

$$= F\left(\frac{1-x}{1+x}\right).$$

$$1-\frac{f'(-x,-x^2)}{f(-x,-x^2)} = \frac{d}{dx}\log\frac{f(-x,-x^2)}{f(x,-x^2)}\Big/ dx$$

$$f(a_1,v_1) = f(u_n,v_n) + u_1 f\left(\frac{u_{n+1}}{u_1},\frac{u_{n+1}}{u_1}\right) + v_1 f\left(\frac{u_{n-1}}{v_1},\frac{v_{n+1}}{v_1}\right)$$

$$+ u_2 f\left(\frac{u_{n-2}}{u_2},\frac{u_{n+2}}{u_2}\right) + v_2 f\left(\frac{u_{n+2}}{v_2},\frac{v_{n+2}}{v_2}\right) + \&c$$

If $ab = cd$ then

$$f(a,b)f(c,d)f\left(ax,\frac{b}{x}\right)f\left(cx,\frac{d}{x}\right)$$

$$-f(-a,-b)f(-c,-d)f\left(-ax,-\frac{b}{x}\right)f\left(-cx,-\frac{d}{x}\right)$$

$$= 2af\left(\frac{c}{a},ad\right)f\left(\frac{d}{cx},acx\right)f\left(x,\frac{ab}{x}\right)\psi(ab)$$

(Deduction).

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x. $f(p, kq)\, f(q, kp) = f(p,q)\, f(k, k^3)$ where $k = pq$.

xi. $f(p,q) + f(-p,-q) = 2 f(p^3 q, p q^3)$

xii. $f(p,q) - f(-p,-q) = 2p\, f\left(\frac{q}{p}, \frac{p}{q} k^4\right)$ where $k = pq$

xiii. $f(p,q)\, f(-p,-q) = f(-p^2, -q^2)\, f(-pq, -pq)$

xiv. $f^2(p,q) + f^2(-p,-q) = 2 f(p^2, q^2)\, f(pq, pq)$

xv. $f^2(p,q) - f^2(-p,-q) = 4p\, f\left(\frac{q}{p}, \frac{p}{q} k^4\right) f(k^2, k^6)$

where $k = pq$.

N.B. Of all the functions forms of $f(p,q)$, the most important are $f(-x,-x)$, $f(-x,-x^2)$ and $f(-x,-x^3)$

$f(x,x) = \phi(x)$ and $f(x,x^3) = \psi(x)$.

Ex. 1. $f(-x,x) = \phi(-x^4)$

2. $f(x,x^7)\, f(x^3, x^5) = \psi(x)\, \psi(x^4)$

3. $2 f(x^3, x^5) = \psi(\sqrt{x}) + \psi(-\sqrt{x})$

4. $2 f(x, x^7) = \dfrac{\psi(\sqrt{x}) - \psi(-\sqrt{x})}{\sqrt{x}}$.

17. i $\phi(x) = 1 + 2x + 2x^4 + 2x^9 + 2x^{16} + \&c$

$= \frac{1+x}{1-x} \cdot \frac{1-x^2}{1+x^2} \cdot \frac{1+x^3}{1-x^3} \cdot \frac{1-x^4}{1+x^4}$ &c

ii. $\psi(x) = 1 + x + x^3 + x^6 + x^{10} + \&c$

$= \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^3} \cdot \frac{1-x^6}{1-x^5}$ &c

iii $\frac{1}{2} \log \phi(x) = \frac{x}{1+x} + \frac{x^3}{3(1+x^3)} + \frac{x^5}{5(1+x^5)} + \&c$

iv $\log \psi(x) = \frac{x}{1+x} + \frac{x^2}{2(1+x^4)} + \frac{x^3}{3(1+x^3)} + \&c$

v. $\frac{\psi(x)}{\phi(x)} = \frac{1+x^2}{1+x} \cdot \frac{1+x^4}{1+x^3} \cdot \frac{1+x^6}{1+x^5}$ &c.

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$$\phi(x)+\phi(x^4)=\frac{1+2}{(1-x)(1-x^4)(1-x^7)(1-x^8)(1-x^{12})(1-x^{15})(1-x^{17})}$$

$$\phi(x)-\phi(x^4)=$$

If $F(x^2)=t$

then $F\left(\frac{2x}{1+x}\right)=\frac{\sqrt{t}}{1+(1-t)(1-t+\frac{3}{4}(t^2)}$ nearly.

$$\left(x+\frac{x^2}{2}+\frac{21x^3}{64}+\frac{21x^4}{128}+\&c\right)^{11}$$

$$=x^{11}+\frac{11}{2}x^{12}+\frac{1111}{64}x^{13}+\frac{111111}{896}+\frac{111111}{2688}x^{14}+\&c$$

θ is less than $\frac{1}{2160}\left\{\frac{4x}{8+\cdot14(\frac{4}{3}x)^2}\right\}^5$.

or $F(1-e^{-x})=\frac{x}{10+\sqrt{36+x^2}}-\frac{1}{2160}\cdot\left(\frac{x}{8+\cdot14x^2}\right)^5$.

$$\frac{1}{\phi(x^4)}=\frac{1}{\phi(x)\pm\phi(x^2)}+\frac{1}{\phi(-x)\pm\phi(x^2)}$$

$$\pm\frac{1}{\phi(-x^4)}=\frac{1}{\phi(x)\pm\phi(-x^2)}+\frac{1}{\phi(-x)\pm\phi(-x^2)}$$

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vi $\phi(x) + \phi(-x) = 2\phi(x^4)$

vii $\phi(x) - \phi(-x) = 4x\,\psi(x^8)$

viii $\phi(x)\,\phi(-x) = \phi^2(-x^2)$.

ix $\phi(x)\,\psi(x^2) = \psi^2(x)$

x. $\phi^2(x) - \phi^2(-x) = 8x\,\psi^2(x^4)$

xi. $\phi^2(x) + \phi^2(-x) = 2\,\phi^2(x^2)$

xii. $\phi^4(x) - \phi^4(-x) = 16x\,\psi^4(x^2)$.

xiii. $\psi^2(x) + \psi^2(-x) = 2\,\psi(x^2)\,\phi(x^4)$

xiv. If $\left(\frac{1-z}{1+z}\right)^2 = \left\{\frac{\phi(-x)}{\phi(x)}\right\}^4$ then $1 - z^2 = \left\{\frac{\phi(-x^2)}{\phi(x^2)}\right\}^4$

Ex. 1. $\frac{\psi(x)}{\psi(-x)} = \sqrt{\frac{\phi(x)}{\phi(-x)}}$.

2. $\psi(x)\,\psi(-x) = \psi(x^2)\,\phi(-x^2)$

3. $\frac{\psi(x)\,\psi(-x)}{\psi(x^2)\,\psi(-x^2)} = \frac{\psi(-x^2)}{\psi(x^4)}$.

18. If $F(x) = e^{-\pi \cdot \frac{1 + (\frac{1}{2})^2(1-x) + \left(\frac{1\cdot3}{2\cdot4}\right)^2(1-x)^2 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2(1-x)^3 + \&c}{1 + (\frac{1}{2})^2 x + \left(\frac{1\cdot3}{2\cdot4}\right)^2 x^2 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 x^3 + \&c}}$

i. $F(x) = \frac{x}{16}\, e^{4 \cdot \frac{(\frac{1}{2})^2 \frac{1}{1\cdot2} x + \left(\frac{1\cdot3}{2\cdot4}\right)^2 \left(\frac{1}{1\cdot2} + \frac{1}{3\cdot4}\right) x^2}{1 + (\frac{1}{2})^2 x + \left(\frac{1\cdot3}{2\cdot4}\right)^2 x^2 + \&c}}$

ii. $F\left(1 - \frac{1}{x}\right) + \theta = \frac{\log_e x}{10 + \sqrt{36 + (\log_e x)^2}}$ where θ is numerically much less than $\frac{2}{135} F^{-5}\left(1 - \frac{1}{x}\right)$. $\theta = \frac{1}{2160} \cdot \left\{\frac{\log_e x}{8 + (\log_e x)^2}\right\}^5$.

iii. $\log_e F(x)\, \log_e F(1-x) = \pi^2$

iv $F(1-x) + F\left(1 - \frac{1}{x}\right) = 0$

v. $F\left\{\frac{4x}{(1+x)^2}\right\} = \sqrt{F(x^2)}$

N. B. If we know the expansion of $F\left(\frac{2x}{1+x}\right)$ to n terms, then we can find its expansion to $2n$ terms as follows — Suppose we know the expansion of $F\left(\frac{2x}{1+x}\right)$ to n terms.

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$$2\frac{\psi'(x)}{\psi(x)} - 2x\frac{\psi'(x^2)}{\psi(x^2)} = \frac{\phi'(x)}{\phi(x)}$$

$$\frac{\phi'(-x)}{\phi(-x)} - 4x\frac{\phi'(-x^2)}{\phi(-x^2)} = \frac{\phi'(x)}{\phi(x)}.$$

If $F(x) = z$ & $y = \sqrt{1+\left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \text{&c}}$, then

$\phi(z) = y$; $\phi(-z) = y\sqrt{1-x}$.

$\phi(z^2) = y\sqrt{\frac{1+\sqrt{1-x}}{2}}$; $\phi(-z^2) = y\sqrt[8]{1-x}$.

$\phi(z^4) = y\cdot\frac{1+\sqrt[4]{1-x}}{2}$.

$\phi(\sqrt{z}) = y\sqrt{1+\sqrt{x}}$; $\phi(-\sqrt{z}) = y\sqrt{1-\sqrt{x}}$.

$\phi(\sqrt[4]{z}) = y\left(1+\sqrt[4]{x}\right)$; $\phi(-\sqrt[4]{z}) = y\left(1-\sqrt[4]{x}\right)$.

$\psi(z) = \frac{y}{\sqrt{2}}\sqrt[8]{\frac{x}{2}}$.

$\psi(z^2) = \frac{y}{2}\sqrt[4]{\frac{x}{2}}$.

$\psi(z^4) = \frac{y}{2}\sqrt{\frac{1-\sqrt{1-x}}{2^2}}$.

$\psi(z^8) = y\cdot\frac{1-\sqrt[4]{1-x}}{4^2}$.

$\psi(\sqrt{z}) = y\sqrt[4]{\frac{1+\sqrt{x}}{2}}\cdot\sqrt[16]{\frac{x}{2}}$.

$\psi(\sqrt[4]{z}) = y\sqrt{1+\sqrt[4]{x}}\cdot\sqrt[8]{\frac{1+\sqrt{x}}{2}}\cdot\sqrt[32]{\frac{x}{2}}\ldots$

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Writing $\frac{x^2}{2-x^2}$ for x we have the expansion of $\Phi(x^2)$ to $2n$ terms i.e that of $\left[\Phi\left\{\frac{4x}{(1+x)^2}\right\}\right]^2$ to $2n$ terms. Extracting the Square root and expanding the result in ascending powers of $\frac{2x}{1+x^2}$ we can find the expansion of $\Phi\left(\frac{2x}{1+x}\right)$ to $2n$ terms.

VI. $2\Phi\left(\frac{2x}{1+x}\right) = x + \frac{5}{16}x^3 + \frac{369}{2048}x^5 + \frac{4097}{32768}x^7 + \frac{1594895}{16777216}x^9 + \&c$

VII. $2F(1-e^{-8x}) = x - \frac{x^3}{3} + \frac{31}{120}x^5 - \frac{661}{2520}x^7 + \frac{219677}{725760}x^9 - \&c$

VIII. $2F\left(1-e^{-\frac{8x}{1-x^2}}\right) = x + \frac{2}{3}x^3 + \frac{31}{120}x^5 + \frac{37}{1260}x^7 + \frac{5981}{725760}x^9 + \&c$

Ex. 1. $\Phi(0)=0$, $F\left(\frac{1}{2}\right) = e^{-\pi}$, $F(1)=1$, $F\left\{(\sqrt{2}-1)^4\right\} = e^{-\pi\sqrt{2}}$

IX. $\phi^2(x) = 1 + \left(\frac{1}{2}\right)^2\left\{1 - \frac{\phi^4(-x)}{\phi^4(x)}\right\} + \left(\frac{1.3}{2.4}\right)^2\left\{1 - \frac{\phi^4(-x)}{\phi^4(x)}\right\}^2 + \&c$

X. $1 + \left(\frac{1}{2}\right)^2\left\{\frac{\phi^4(-x)}{\phi^4(x)}\right\} + \left(\frac{1.3}{2.4}\right)^2\left\{\frac{\phi^4(-x)}{\phi^4(x)}\right\}^2 + \&c$

$= \frac{\phi^2(x)}{n\phi^2(x^n)}\left\{1 + \left(\frac{1}{2}\right)^2\left[\frac{\phi^4(-x^n)}{\phi^4(x^n)}\right] + \left(\frac{1.3}{2.4}\right)^2\left[\frac{\phi^4(-x^n)}{\phi^4(x^n)}\right]^2 + \&c\right\}$

XI. $F\left\{\frac{\phi^4(-x^n)}{\phi^4(x^n)}\right\} = \sqrt[n]{F\left\{\frac{\phi^4(-x)}{\phi^4(x)}\right\}}$

XII. $F\left\{1 - \frac{\phi^4(-x)}{\phi^4(x)}\right\} = \sqrt[n]{F\left\{1 - \frac{\phi^4(-x^n)}{\phi^4(x^n)}\right\}}$

XIII. $F\left\{1 - \frac{\phi^4(-x)}{\phi^4(x)}\right\} = x$

XIV. $\phi^2\{F(x)\} = 1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 x^3 + \&c$

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$$\begin{cases} 1+2e^{-\pi}+2e^{-4\pi}+2e^{-9\pi}+\&c = \frac{\sqrt[4]{\pi}}{\Gamma\frac{1}{4}} = k \\ 1-2e^{-\pi}+2e^{-4\pi}-2e^{-9\pi}+\&c = \frac{k}{\sqrt[4]{2}} \end{cases}$$

$$\begin{cases} 1+2(e^{-\pi})^{\frac{1}{2}}+2(e^{-4\pi})^{\frac{1}{2}}+2(e^{-9\pi})^{\frac{1}{2}}+\&c = \frac{k}{2}\sqrt{2+\sqrt{2}} \\ 1-2(e^{-\pi})^{\frac{1}{2}}+2(e^{-4\pi})^{\frac{1}{2}}-2(e^{-9\pi})^{\frac{1}{2}}+\&c = \frac{k}{\sqrt[4]{2}} \end{cases}$$

$$\begin{cases} 1+2(e^{-\pi})^{\frac{1}{4}}+2(e^{-4\pi})^{\frac{1}{4}}+2(e^{-9\pi})^{\frac{1}{4}}+\&c = \frac{k}{2}\left(1+\frac{1}{\sqrt[4]{2}}\right) \\ 1-2(e^{-\pi})^{\frac{1}{4}}+2(e^{-4\pi})^{\frac{1}{4}}-2(e^{-9\pi})^{\frac{1}{4}}+\&c = \frac{k}{\sqrt[4]{2}}\sqrt[4]{\sqrt{2}+\sqrt{2}} \end{cases}$$

$$\begin{cases} 1+2\sqrt{e^{-\pi}}+2\sqrt{e^{-4\pi}}+2\sqrt{e^{-9\pi}}+\&c = \frac{k}{\sqrt[4]{2}}\sqrt{1+\sqrt{2}} \\ 1-2\sqrt{e^{-\pi}}+2\sqrt{e^{-4\pi}}-2\sqrt{e^{-9\pi}}+\&c = \frac{k}{\sqrt[4]{2}}\sqrt{\sqrt{2}-1} \end{cases}$$

$$\begin{cases} 1+2\sqrt[4]{e^{-\pi}}+2\sqrt[4]{e^{-4\pi}}+2\sqrt[4]{e^{-9\pi}}+\&c = k\left(1+\frac{1}{\sqrt[4]{2}}\right) \\ 1-2\sqrt[4]{e^{-\pi}}+2\sqrt[4]{e^{-4\pi}}-2\sqrt[4]{e^{-9\pi}}+\&c = k\left(1-\frac{1}{\sqrt[4]{2}}\right) \end{cases}$$

$$\begin{cases} 1+e^{-\pi}+e^{-3\pi}+e^{-6\pi}+e^{-10\pi}+\&c = k\sqrt[8]{\frac{e^{\pi}}{32}} \\ 1+(e^{-\pi})^{\frac{1}{2}}+(e^{-3\pi})^{\frac{1}{2}}+(e^{-6\pi})^{\frac{1}{2}}+(e^{-10\pi})^{\frac{1}{2}}+\&c = k\sqrt[4]{\frac{e^{\pi}}{32}} \end{cases}$$

$$\begin{cases} 1+(e^{-\pi})^{\frac{1}{4}}+(e^{-3\pi})^{\frac{1}{4}}+(e^{-6\pi})^{\frac{1}{4}}+(e^{-10\pi})^{\frac{1}{4}}+\&c = \frac{k}{4}\sqrt{e^{\pi}(2-\sqrt{2})} \\ 1+(e^{-\pi})^{8}+(e^{-3\pi})^{8}+(e^{-6\pi})^{8}+\&c = \frac{k}{4}e^{\pi}\left(1-\frac{1}{\sqrt{2}}\right) \end{cases}$$

$$\begin{cases} 1+\sqrt{e^{-\pi}}+\sqrt{e^{-3\pi}}+\sqrt{e^{-6\pi}}+\&c = k\sqrt[4]{\frac{1+\frac{1}{\sqrt{2}}}{2\sqrt{2}}}\sqrt[4]{e^{\pi}} \\ 1+\sqrt[4]{e^{-\pi}}+\sqrt[4]{e^{-3\pi}}+\sqrt[4]{e^{-6\pi}}+\&c = k\sqrt[32]{\frac{e^{\pi}}{2}}\sqrt[8]{1+\sqrt{2}}\sqrt{\frac{1+\frac{1}{\sqrt{2}}}{\sqrt{2}}} \end{cases}$$

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19. i.e. If $y = \pi \dfrac{1 + (\frac{1}{2})^2(1-x) + (\frac{1\cdot3}{2\cdot4})^2(1-x)^2 + \&c}{1 + (\frac{1}{2})^2 x + (\frac{1\cdot3}{2\cdot4})^2 x^2 + \&c}$, then

$$1 + (\tfrac{1}{2})^2 x + (\tfrac{1\cdot3}{2\cdot4})^2 x^2 + (\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6})^2 x^3 + \&c$$
$$= (1 + 2e^{-\pi} + 2e^{-4\pi} + 2e^{-9\pi} + 2e^{-16\pi} + \&c)^2$$

Cor. If $\sqrt{\alpha}\,\beta = \sqrt{\pi}$, then

$$\sqrt{\alpha}\left\{\tfrac{1}{2} + e^{-\alpha^2} + e^{-4\alpha^2} + e^{-9\alpha^2} + \&c.\right\}$$
$$= \sqrt{\beta}\left\{\tfrac{1}{2} + e^{-\beta^2} + e^{-4\beta^2} + e^{-9\beta^2} + \&c.\right\}$$

20. If $\alpha\beta = \pi$, then

$$\sqrt{\alpha}\left\{1 + \frac{2}{(1+\alpha^2)^{n+1}} + \frac{2}{(1+4\alpha^2)^{n+1}} + \frac{2}{(1+9\alpha^2)^{n+1}} + \&c\right\}$$
$$= \frac{\lfloor n-\frac{1}{2}}{\lfloor n}\sqrt{\beta}\left\{1 + 2e^{-2\beta}\phi(2\beta) + 2e^{-4\beta}\phi(4\beta) + \&c\right\}.$$

where $\phi(\alpha) = 1 + \frac{n}{1}\cdot\frac{x}{1!} + \frac{n(n-1)}{n(n-\frac{1}{2})}\frac{x^2}{\lfloor 2} + \frac{n(n-1)(n-2)}{n(n-\frac{1}{2})(n-1)}\frac{x^3}{\lfloor 3} + \&c$

to $n+1$ terms.

$$= \frac{1}{\lfloor 2n}\left\{(2x)^n \lfloor n + \frac{n}{1}(2x)^{n-1}\lfloor n+1 + \frac{n(n-1)}{\lfloor 2}(2x)^{n-2}\lfloor n+2 + \&c \text{ to infinity.}\right\}$$

Cor. If $\alpha\beta = \pi$, then

$$\sqrt{\alpha}\left\{\tfrac{1}{2} + e^{-\alpha^2} + e^{-4\alpha^2} + e^{-9\alpha^2} + \&c\right\}$$
$$= \sqrt{\beta}\left\{\tfrac{1}{2} + e^{-\beta^2} + e^{-4\beta^2} + e^{-9\beta^2} + \&c\right\}$$

Ex. 1. $(1 + 2e^{-\pi} + 2e^{-4\pi} + 2e^{-9\pi} + \&c)^2$

$$= 1 + (\tfrac{1}{2})^2\tfrac{1}{2} + (\tfrac{1\cdot3}{2\cdot4})^2\tfrac{1}{2^2} + (\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6})^2\tfrac{1}{2^3} + \&c = \frac{\sqrt{\pi}}{(\lfloor -\frac{1}{2})^2}$$

2. $\frac{1}{2} + e^{-\pi\sqrt{2}} + e^{-4\pi\sqrt{2}} + e^{-9\pi\sqrt{2}} + \&c = \dfrac{\lfloor\frac{1}{2}}{\sqrt{\pi}\,\lfloor\frac{1}{4}}$

3. $\dfrac{\pi - \frac{1}{4}}{e^{\pi}} + \dfrac{4\pi - \frac{1}{4}}{e^{4\pi}} + \dfrac{9\pi - \frac{1}{4}}{e^{9\pi}} + \&c = \dfrac{1}{8}$.

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$$\begin{cases} (1-e^{-\pi})(1-e^{-2\pi})(1-e^{-3\pi})(1-e^{-4\pi})\,\&c = \frac{1}{2}\,\frac{\sqrt[24]{e^{\pi}}}{\sqrt[8]{\ }} \\ (1-e^{-2\pi})(1-e^{-4\pi})(1-e^{-6\pi})(1-e^{-8\pi})\,\&c = \frac{1}{\sqrt{2}}\,\sqrt[12]{e^{\pi}}. \\ (1-e^{-4\pi})(1-e^{-8\pi})(1-e^{-12\pi})\,\&c = \frac{1}{2}\sqrt{2}\,\sqrt[6]{e^{\pi}}. \\ (1-e^{-8\pi})(1-e^{-11\pi})(1-e^{-24\pi})\,\&c = \frac{1}{2}\sqrt[3]{e^{\pi}}\,\sqrt[4]{\tfrac{1}{2}(\sqrt{2}-\tfrac{1}{\sqrt{2}})}. \end{cases}$$

$$\begin{cases} (1-e^{-\pi})(1-e^{-3\pi})(1-e^{-5\pi})\,\&c = \dfrac{\sqrt[8]{2}}{\sqrt[24]{e^{\pi}}} \\ (1-e^{-2\pi})(1-e^{-6\pi})(1-e^{-10\pi})\,\&c = \dfrac{\sqrt[8]{\ }}{\sqrt[12]{e^{\pi}}} \\ (1-e^{-4\pi})(1-e^{-12\pi})(1-e^{-20\pi})\,\&c = \dfrac{\sqrt[4]{2(2+\sqrt{2})}\,\sqrt[8]{2}}{\sqrt[6]{e^{\pi}}}. \end{cases}$$

$$\begin{cases} (1+e^{-\pi})(1+e^{-3\pi})(1+e^{-5\pi})\,\&c = \dfrac{\sqrt[8]{2}}{\sqrt[24]{e^{\pi}}}. \\ (1+e^{-2\pi})(1+e^{-6\pi})(1+e^{-10\pi})\,\&c = \dfrac{\sqrt[4]{(1+\sqrt{2})}\,\sqrt[8]{2}}{\sqrt[12]{e^{\pi}}}. \end{cases}$$

$$\left(\tfrac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(1+\tfrac{1}{3}\right)x^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(1+\tfrac{1}{3}+\tfrac{1}{5}\right)x^3 + \&c$$

$$= -\tfrac{1}{2}\left\{1+\left(\tfrac{1}{2}\right)^2 x + \&c\right\}\log(1-x).$$

Riemann.

$$\left(\tfrac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(1+\tfrac{1}{2}\right)x^2 + \&c$$

$$= -2\left\{1+\left(\tfrac{1}{2}\right)^2 x + \&c\right\}\left\{\log\sqrt{\pi} + \left(\tfrac{1}{2}\right)^2 x + \&c\right]$$

$$+ \tfrac{1}{3}\log(-x) - \tfrac{1}{12}\log x - \frac{\pi}{12}\left[1+\left(\tfrac{1}{2}\right)^2(1-x) + \&c\right]\Big\}$$

$$= 4\left(\frac{1}{e^{\omega_1}} + \frac{1}{2}\cdot\frac{1}{e^{2\omega_1}} + \&c\right)$$

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CHAPTER XVI

1. $h\phi(h) + h\phi(2h) + h\phi(3h) + h\phi(4h) + \&c$

$= \int_0^\infty \phi(x)\,dx + F(h)$ where $F(0) = 0$

$F(h)$ can be found by expanding the left but writing the constant instead of a Series.

Cor. If $h\phi(h) = ah^p + bh^q + ch^r + dh^s + \&c$

and if p, q, r, s &c be not negative, then

$h\phi(h) + h\phi(2h) + h\phi(3h) + h\phi(4h) + \&c$

$= \int_0^\infty \phi(x)\,dx + a\frac{\beta_p}{p} h^p \cos\frac{\pi p}{2} + b\frac{\beta_q}{q} h^q \cos\frac{\pi q}{2} + \&c$

N.B. $F(h)$ is not a terminating Series; but if p, q, r, s &c be odd integers, $F(h)$ appears to be 0 or to have some finite value: In this case $F(h)$ is really a function of $e^{-\frac{\pi}{h}}$ which is 0 when $h=0$ but which can't be expressed as a function in ascending powers of h. When h is small the difference between the real and the apparent value of $F(h)$ is so small that we can neglect it. e.g.

The apparent value of $x + \frac{2x}{1+x^2} + \frac{2x}{1+(2x)^2} + \&c$

$= \pi$, the real value being $\pi\left(1 + 2e^{-\frac{2\pi}{x}} + 2e^{-\frac{4\pi}{x}} + \&c\right)$

Ex. Shew that $x\left\{\phi(0) + [\phi(x) + \phi(-x)] + [\phi(2x) + \phi(-2x)] + \&c\right\}$

$= \int_{-\infty}^{\infty} \phi(x)\,dx + F(e^{-\frac{\pi}{x}})$ where $F(0) = 0$.

2. $\frac{1^{n-1}}{e^{1^n x}} + \frac{2^{n-1}}{e^{2^n x}} + \frac{3^{n-1}}{e^{3^n x}} + \frac{4^{n-1}}{e^{4^n x}} + \&c$

$= \frac{1}{m}\cdot\frac{\lfloor\frac{m}{n}}{x^{\frac{m}{n}}} + \frac{B_m}{m}\cos\frac{\pi m}{2} - \frac{x}{\lfloor 1}\cdot\frac{B_{m+n}}{m+n}\cos\frac{\pi(m+n)}{2}$

$+ \frac{x^2}{\lfloor 2}\cdot\frac{B_{m+2n}}{m+2n}\cos\frac{\pi(m+2n)}{2} - \&c$

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$$\frac{1^m \cdot 1^n}{e^x - 1} + \frac{2^m (2^n + 1^n)}{e^{2x} - 1} + \frac{3^m (3^n + 1^n)}{e^{3x} - 1} + \frac{4^m (4^n + 2^n + 1^n)}{e^{4x} - 1} + \&c$$

$$= \frac{\lfloor m}{x^{m+1}} S_{m+1} S_{m-n+1} + \frac{\lfloor n}{x^{n+1}} S_{n+1} S_{n-m+1}$$

$$+ \frac{1}{2} S_{1-m} S_{1-n} - \frac{1}{2} S_{-m} S_{-n} + \frac{13}{12} x S_{-1-m} S_{-1-n}$$

$$- \frac{13}{16} x^3 S_{-3-m} S_{-3-n} + \&c$$

$$1^m \{1^n e^{-x} + 2^n e^{-2x} + 3^n e^{-3x} + \&c\}$$
$$+ 2^m \{1^n e^{-2x} + 2^n e^{-4x} + 3^n e^{-6x} + \&c\}$$
$$+ 3^m \{1^n e^{-3x} + 2^n e^{-6x} + 3^n e^{-9x} + \&c\}$$
$$+ \&c \quad \&c \quad \&c \quad \&c \quad \&c$$

$$= \frac{\lfloor m}{x^{m+1}} S_{1+n-m} + \frac{\lfloor n}{x^{n+1}} S_{1+n-m}$$

$$+ S_{-m} S_{-n} - \frac{x}{12} \cdot S_{-m-1} S_{-n-1} + \frac{x^2}{12} S_{-m-2} S_{-n-2} \&c$$

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N. B. This is true only when n is positive.

3. $\frac{1^{l-1}}{(1+x^n)^m} + \frac{2^{l-1}}{(1+2^n x^n)^m} + \frac{3^{l-1}}{(1+3^n x^n)^m} + \&c$

$= \frac{\lfloor \frac{l}{n} \rfloor m - \frac{l}{n} - 1}{l x^l \lfloor m-1} + \frac{B_l}{l} \cos\frac{\pi l}{2} - \frac{m}{\lfloor 1} . x^n \frac{B_{l+n}}{l+n} \cos\frac{\pi(l+n)}{2}$

$+ \frac{m(m+1)}{\lfloor 2} . x^{2n} \frac{B_{l+2n}}{l+2n} \cos\frac{\pi(l+2n)}{2} - \&c$

4. $\frac{1^{m-1}}{e^{1^n x} - 1} + \frac{2^{m-1}}{e^{2^n x} - 1} + \frac{3^{m-1}}{e^{3^n x} - 1} + \&c$

$= \frac{1}{m} . \frac{\lfloor \frac{m}{n}}{x^{\frac{m}{n}}} S_{\frac{m}{n}} + \frac{1}{2} S_{1+n-m} - \frac{1}{2} \frac{B_m}{m} \cos\frac{\pi}{2} m$

$+ \frac{x}{\lfloor 1} . \frac{B_2}{2} . \frac{B_{m+n}}{m+n} \cos\frac{\pi(m+n)}{2} - \frac{x^3}{\lfloor 3} . \frac{B_4}{4} . \frac{B_{m+3n}}{m+3n} \cos\frac{\pi(m+3n)}{2}$

$+ \frac{x^5}{\lfloor 5} . \frac{B_6}{6} . \frac{B_{m+5n}}{m+5n} \cos\frac{\pi(m+5n)}{2} - \&c$

Cor. $\frac{1^{n-1}}{e^{1^n x} - 1} + \frac{2^{n-1}}{e^{2^n x} - 1} + \frac{3^{n-1}}{e^{3^n x} - 1} + \&c$

$= \frac{C_0 - \frac{1}{n} \log x}{x} - \frac{1}{2} . \frac{B_n}{n} \cos\frac{\pi n}{2} + \frac{x}{\lfloor 1} \frac{B_2}{2} . \frac{B_{2n}}{2n} \cos\frac{2\pi n}{2}$

$- \frac{x^3}{\lfloor 3} . \frac{B_4}{4} . \frac{B_{4n}}{4n} \cos 2\pi n + \frac{x^5}{\lfloor 5} \frac{B_6}{6} . \frac{B_{6n}}{6n} \cos 3\pi n - \&c$

5. If $\phi(t) = \frac{1^{m-1}}{(e^{1^p x})^t - q} + \frac{2^{m-1}}{(e^{2^p x})^t - q} + \frac{3^{m-1}}{(e^{3^p x})^t - q} + \&c$, the

$1^{n-1}\phi(1) + 2^{n-1}\phi(2) + 3^{n-1}\phi(3) + 4^{n-1}\phi(4) + \&c$

$= \left\{ \frac{\lfloor \frac{m}{p}}{m x^{\frac{m}{p}}} S_{1 + \frac{m}{p} q - n} \right\} + \left\{ \frac{\lfloor \frac{n}{q}}{n x^{\frac{n}{q}}} S_{1 + \frac{n}{q} p - m} \right\}$

$+ \frac{B_m}{m} . \frac{B_n}{n} \cos\frac{\pi m}{2} \cos\frac{\pi n}{2} - \frac{x}{\lfloor 1} . \frac{B_{m+p}}{m+p} . \frac{B_{n+q}}{n+q} \cos\frac{\pi(m+p)}{2} \cos\frac{\pi(n+q)}{2}$

$+ \frac{x^2}{\lfloor 2} . \frac{B_{m+2p}}{m+2p} \frac{B_{n+2q}}{n+2q} \cos\frac{\pi(m+2p)}{2} \cos\frac{\pi(n+2q)}{2} - \&c$

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$$\psi(x) - x\psi(x^9) = \frac{\phi(-x^9)}{\sqrt[3]{\phi(-x^3)}}\sqrt[3]{\psi(x^3)}.$$

$$\{3\phi(-x^9) - \phi(-x)\}^3 = 8\,\frac{\psi^3(x)}{\psi(x^3)}\,\phi(-x^3)$$

$$3617 + 16320\left(\frac{1^{15}x}{1-x} + \frac{2^{15}x^2}{1-x^2} + \frac{3^{15}x^3}{1-x^3} + \dots\right)$$
$$1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \dots\right)$$
$$= 1617\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \dots\right)\right\}^3$$
$$+ 2000\left\{1 - 504\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \dots\right)\right\}^2$$

$$43867 - 28728\left(\frac{1^{17}x}{1-x} + \frac{2^{17}x^2}{1-x^2} + \frac{3^{17}x^3}{1-x^3} + \dots\right)$$
$$1 - 504\left(\frac{1^{17}x}{1-x} + \frac{2^{17}x^2}{1-x^2} + \dots\right)$$
$$= 38367\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \dots\right)\right\}^3$$
$$+ 5500\left\{1 - 504\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \dots\right)\right\}^2$$

$$174611 + 13200\left(\frac{1^{19}x}{1-x} + \frac{2^{19}x^2}{1-x^2} + \dots\right)$$
$$1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \dots\right)$$
$$= 53361\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \dots\right)\right\}^3$$
$$+ 121250\left\{1 - 504\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \dots\right)\right\}^2$$

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Cor. If $\frac{m}{p} = \frac{n}{q} = k$, R.H.S becomes

$$\frac{\lfloor k}{mnx^k}\left\{k\left(\Sigma\frac{1}{k-1} - C_0 - \log_e x\right) + C_0(m+n)\right\} + \frac{\beta_m}{m}\cdot\frac{\beta_n}{n}\cos\frac{\pi m}{2}\cos\frac{\pi n}{2} \ldots$$

6. $(1+2r\cos\theta+r^2)(1+2r^3\cos\theta+r^6)(1+2r^5\cos\theta+r^{10})$ &c

$\times(1-r^2)(1-r^4)(1-r^6)(1-r^8)$ &c

$= 1+2r\cos\theta+2r^4\cos 2\theta+2r^9\cos 3\theta+2r^{16}\cos 4\theta+$ &c

Cor. 1. $(1-r^3)(1-r^9)(1-r^{15})(1-r^{21})$ &c

$$= \frac{1+r-(r^4+2r^9+r^{16})+(r^{25}+2r^{36}+r^{49})-\&c}{1+r+r^3+r^6+r^{10}+r^{15}+r^{21}+\&c}$$

$$= \frac{1+r\omega-(r^4\omega+2r^9+r^{16}\omega)+(r^{25}\omega+2r^{36}+r^{49}\omega)-\&c}{1+r\omega+r^3+r^6+r^{10}\omega+r^{15}+r^{21}+r^{28}\omega+\&c}$$

$$= \frac{1-r^3-r^{15}+r^{24}+r^{48}-\&c}{1+r^9+r^{27}+r^{54}+\&c}$$

$$= \frac{1-2r^9+2r^{36}-2r^{81}+\&c}{1+r^3+r^6+r^{15}+r^{21}+\&c}$$

2. If $(1+r\sqrt{x}+r^2)(1+r^3\sqrt{x}+r^6)(1+r^5\sqrt{x}+r^{10})$ &c

$\times(1-r^2)(1-r^4)(1-r^6)(1-r^8)(1-r^{10})$ &c $= \phi(x)$

Find $\phi(1)$, $\phi(2)$, $\phi(3)$ and $\phi(4)$ in ascending powers of

7. If $\alpha\beta = \pi^2$, then

$\sqrt[4]{\alpha e^{-\frac{\alpha}{3}}}(1-e^{-2\alpha})(1-e^{-4\alpha})(1-e^{-6\alpha})$ &c

$= \sqrt[4]{\beta e^{-\frac{\beta}{3}}}(1-e^{-2\beta})(1-e^{-4\beta})(1-e^{-6\beta})$ &c

8. If $\alpha\beta = \pi^2$

$\frac{\alpha}{12} - \frac{1}{4}\log_e\alpha + \frac{1}{1(e^{2\alpha}-1)} + \frac{1}{2(e^{4\alpha}-1)} + \frac{1}{3(e^{6\alpha}-1)}$ &c

$= \frac{\beta}{12} - \frac{1}{4}\log_e\beta + \frac{1}{1(e^{2\beta}-1)} + \frac{1}{2(e^{4\beta}-1)} \ldots$

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$$1 + 480\left(\frac{1^7 x}{1-x} + \frac{2^7 x^2}{1-x^2} + \frac{3^7 x^3}{1-x^3} + \&c\right)$$

$$= \left\{1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \&c\right)\right\}^2.$$

$$\left\{1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \&c\right)\right\}$$

$$\times \left\{1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{1-x^2} + \frac{3^5 x^3}{1-x^3} + \&c\right)\right\}$$

$$= 1 - 264\left(\frac{1^9 x}{1-x} + \frac{2^9 x^2}{1-x^2} + \frac{3^9 x^3}{1-x^3} + \&c\right).$$

$$\left\{1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \&c\right)\right\}$$

$$\times \left\{1 - 264\left(\frac{1^9 x}{1-x} + \frac{2^9 x^2}{1-x^2} + \frac{3^9 x^3}{1-x^3} + \&c\right)\right\}$$

$$= 1 - 24\left(\frac{1^{13} x}{1-x} + \frac{2^{13} x^2}{1-x^2} + \frac{3^{13} x^3}{1-x^3} + \&c\right)$$

$$691 + 240.273\left(\frac{1^{11} x}{1-x} + \frac{2^{11} x^2}{1-x^2} + \frac{3^{11} x^3}{1-x^3} + \&c\right)$$

$$= 441\left\{1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \&c\right)\right\}^3$$

$$+ 250\left\{1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{1-x^2} + \frac{3^5 x^3}{1-x^3} + \&c\right)\right\}^2.$$

~~If $S_{2n} = \frac{B_{2n}}{4n}\cos\pi n + \frac{1^{2n-1}x}{1-x} + \frac{2^{2n-1}x^2}{1-x^2} + \&c$ then it is possible to find S_{2n} in the series containing $S_4 S_{2n-4}$, $S_6 S_{2n-6}$ &c and not containing $S_2 S_{2n-2}$.~~

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9. If $\alpha\beta = \pi^2$,

$$\frac{1}{2} + \frac{2\alpha}{e^{2\alpha}-1} + \frac{4\alpha}{e^{4\alpha}-1} + \frac{6\alpha}{e^{6\alpha}-1} + \frac{8\alpha}{e^{8\alpha}-1} + \&c$$
$$+ \frac{2\beta}{e^{2\beta}-1} + \frac{4\beta}{e^{4\beta}-1} + \frac{6\beta}{e^{6\beta}-1} + \frac{8\beta}{e^{8\beta}-1} + \&c$$
$$= \frac{\alpha+\beta}{12}.$$

Cor. $$\frac{2\pi}{e^{2\pi}-1} + \frac{4\pi}{e^{4\pi}-1} + \frac{6\pi}{e^{6\pi}-1} + \&c = \frac{\pi}{12} - \frac{1}{4}.$$

10. If $\alpha\beta = \pi$

$$e^{\frac{\pi^2}{4}}\sqrt{\alpha}\left\{\frac{1}{2} + e^{-\alpha^2}\cos\pi\alpha + e^{-4\alpha^2}\cos 2\pi\alpha + e^{-9\alpha^2}\cos 3\pi\alpha + \&c\right.$$
$$= \sqrt{\beta}\left\{\frac{1}{2} + e^{-\beta^2}\cosh\pi\beta + e^{-4\beta^2}\cosh 2\pi\beta + e^{-9\beta^2}\cosh 3\pi\beta + \&c\right.$$

11. If $\alpha\beta = \pi$

$$e^{\pi^2}\cdot\frac{\sin\pi\alpha\,(1-2e^{-2\alpha^2}\cos 2\pi\alpha + e^{-4\alpha^2})(1-2e^{-4\alpha^2}\cos 4\pi\alpha + e^{-8\alpha^2})\ldots}{\sinh\pi\beta\,(1-2e^{-2\beta^2}\cosh 2\pi\beta + e^{-4\beta^2})(1-2e^{-4\beta^2}\cosh 4\pi\beta + e^{-8\beta^2})\ldots}$$
$$= e^{\frac{\alpha^2-\beta^2}{6}}.$$

12. If $\alpha\beta = \pi$

$$\left\{\frac{\alpha^2}{12} + \frac{\cos 2\pi\alpha}{1(e^{2\alpha}-1)} + \frac{\cos 4\pi\alpha}{2(e^{4\alpha}-1)} + \frac{\cos 6\pi\alpha}{3(e^{6\alpha}-1)} + \&c\right\}$$
$$-\left\{\frac{\beta^2}{12} + \frac{\cosh 2\pi\beta}{1(e^{2\beta}-1)} + \frac{\cosh 4\pi\beta}{2(e^{4\beta}-1)} + \frac{\cosh 6\pi\beta}{3(e^{6\beta}-1)} + \&c\right\}$$
$$= \frac{\pi^2}{2} + \frac{1}{2}\log\frac{\sin\pi\alpha}{e\sinh\pi\beta}.$$

13. If $\alpha\beta = \pi$

$$\frac{\pi}{2} + \frac{\alpha\sinh 2\pi\alpha}{e^{2\alpha^2}-1} + \frac{\alpha\sinh 4\pi\alpha}{e^{4\alpha^2}-1} + \frac{\alpha\sinh 6\pi\alpha}{e^{6\alpha^2}-1} + \&c$$

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$$\frac{1^2x}{(1-x)^2} + \frac{2^2x^2}{(1-x^2)^2} + \frac{3^2x^3}{(1-x^3)^2} + \frac{4^2x^4}{(1-x^4)^2} + \&c$$

$$= \frac{1}{288}\left\{1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \&c\right)\right\}$$

$$- \frac{1}{288}\left\{1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right)\right\}^2$$

$$\frac{1^4x}{(1-x)^2} + \frac{2^4x^2}{(1-x^2)^2} + \frac{3^4x^3}{(1-x^3)^2} + \frac{4^4x^4}{(1-x^4)^2} + \&c$$

$$= \frac{1}{720}\left\{1 + 240\left(\frac{x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \&c\right)\right\}$$

$$\times \left\{1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right)\right\}$$

$$- \frac{1}{720}\left\{1 - 504\left(\frac{x}{1-x} + \frac{2^5x^2}{1-x^2} + \frac{3^5x^3}{1-x^3} + \&c\right)\right\}.$$

$$\frac{1^6x}{(1-x)^2} + \frac{2^6x^2}{(1-x^2)^2} + \frac{3^6x^3}{(1-x^3)^2} + \frac{4^6x^4}{(1-x^4)^2} +$$

$$= \frac{1}{1008}\left\{1 + 480\left(\frac{x}{1-x} + \frac{2^7x^2}{1-x^2} + \frac{3^7x^3}{1-x^3} + \&c\right)\right\}$$

$$- \frac{1}{1008}\left\{1 - 504\left(\frac{x}{1-x} + \frac{2^5x^2}{1-x^2} + \frac{3^5x^3}{1-x^3} + \&c\right)\right\}$$

$$\times \left\{1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right)\right\}$$

$$\frac{1^8x}{(1-x)^2} + \frac{2^8x^2}{(1-x^2)^2} + \&c$$

$$= \frac{1}{720}\left\{1 + 480\left(\frac{x}{1-x} + \frac{2^7x^2}{1-x^2} + \frac{3^7x^3}{1-x^3} + \&c\right)\right\}$$

$$\times \left\{1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right)\right\}$$

$$- \frac{1}{720}\left\{1 - 264\left(\frac{x}{1-x} + \frac{2^9x^2}{1-x^2} + \frac{3^9x^3}{1-x^3} + \&c\right)\right\},$$

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$$+\frac{\beta \sin 2n\beta}{e^{2\beta^2}-1}+\frac{\beta \sin 4n\beta}{e^{4\beta^2}-1}+\frac{\beta \sin 6n\beta}{e^{6\beta^2}-1}+\&c.$$

$$=\frac{\alpha \coth n\alpha-\beta \cot n\beta}{4}.$$

14. If $\alpha\beta=\pi^2$ and n is a positive integer >1.

$$\alpha^n\left\{\frac{B_{2n}}{4n}\cos\pi n+\frac{1^{2n-1}}{e^{2\alpha}-1}+\frac{2^{2n-1}}{e^{4\alpha}-1}+\frac{3^{2n-1}}{e^{6\alpha}-1}+\&c\right\}$$

$$=(-\beta)^n\left\{\frac{B_{2n}}{4n}\cos\pi n+\frac{1^{2n-1}}{e^{2\beta}-1}+\frac{2^{2n-1}}{e^{4\beta}-1}+\frac{3^{2n-1}}{e^{6\beta}-1}+\&c\right\}.$$

Cor 1. $\frac{1^5}{e^{2\pi}-1}+\frac{2^5}{e^{4\pi}-1}+\frac{3^5}{e^{6\pi}-1}+\frac{4^5}{e^{8\pi}-1}+\&c=\frac{1}{504}$.

2. $\frac{1^9}{e^{2\pi}-1}+\frac{2^9}{e^{4\pi}-1}+\frac{3^9}{e^{6\pi}-1}+\frac{4^9}{e^{8\pi}-1}+\&c=\frac{1}{264}$

3. $\frac{1^{13}}{e^{2\pi}-1}+\frac{2^{13}}{e^{4\pi}-1}+\frac{3^{13}}{e^{6\pi}-1}+\frac{4^{13}}{e^{8\pi}-1}+\&c=\frac{1}{24}$

15. If $\alpha\beta=\pi^2$ and n any integer

$$(4\alpha)^{1-n}\left\{\frac{1}{2}S_{2n-1}+\frac{1}{1^{2n-1}(e^{4\alpha}-1)}+\frac{1}{2^{2n-1}(e^{4\alpha}-1)}+\&c\right\}$$

$$-(-4\beta)^{1-n}\left\{\frac{1}{2}S_{2n-1}+\frac{1}{1^{2n-1}(e^{2\beta}-1)}+\frac{1}{2^{2n-1}(e^{4\beta}-1)}+\&c\right\}$$

$$=\frac{B_{2n}}{\lfloor 2n}\left\{(-\alpha)^n+\beta^n\right\}+\pi^2\cdot\frac{B_2}{\lfloor 2}\cdot\frac{B_{2n-2}}{\lfloor 2n-2}\left\{(-\alpha)^{n-2}+\beta^{n-2}\right\}$$

$$-\pi^4\cdot\frac{B_4}{\lfloor 4}\cdot\frac{B_{2n-4}}{\lfloor 2n-4}\left\{(-\alpha)^{n-4}+\beta^{n-4}\right\}+\&c$$ the last term being $-\pi^n\frac{B_n}{\lfloor n}\cdot\frac{B_n}{\lfloor n}(-1)^{\frac{n}{2}}$ or $\pi^{n-1}\frac{B_{n-1}}{n-1}\frac{B_{n+1}}{n+1}(-1)^{\frac{n-1}{2}}\{-2\}$ according as n is even or odd.

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If $P_n = \frac{B_n}{2n}(2^n-1)\cos\frac{n\pi x}{L} + \frac{1^{n-1}x}{1+x} - \frac{2^{n-1}x^2}{1+x^2} + \frac{3^{n-1}x^3}{1+x^3} - \&c$

& $Q_n = \frac{\frac{1}{2}E_{n+1}\cos\frac{n\pi x}{L} + \frac{1^n x}{1-x} - \frac{2^n x^2}{1-x^2} + \frac{3^n x^3}{1-x^3} - \&c}{\frac{1}{2}E_1 + \frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^3}{1-x^3} - \&c}$

then $\frac{1}{2}Q_n = 2^n P_n - \frac{(n-1)(n-2)}{\lfloor 4}2^{n-2}P_{n-2}Q_2 +$

$+ \frac{(n-1)(n-2)(n-3)(n-4)}{\lfloor 4}2^{n-4}P_{n-4}Q_4 - \&c$

If $P_n = \frac{B_n}{2n}\cos\frac{n\pi x}{L} + \frac{1^{n-1}x}{1-x} + \frac{2^{n-1}x^2}{1-x^2} + \frac{3^{n-1}x^3}{1-x^3} + \&c$

& $Q_n = \frac{1}{n+1}\cdot\frac{1^{n+1} - 3^{n+1}x + 5^{n+1}x^2 - 7^{n+1}x^3 + \&c}{1 - 3x + 5x^2 - 7x^3 + \&c}$

then $\frac{1}{2}Q_n = -2^n P_n - \frac{(n-1)(n-2)}{\lfloor 4}2^{n-2}P_{n-2}Q_2 -$

$\frac{(n-1)(n-2)(n-3)(n-4)}{\lfloor 4}2^{n-4}P_{n-4}Q_4 - \&c$

$\log_e \frac{(1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})\&c}{1 - 2x\cos\theta + 2x^4\cos 2\theta - 2x^9\cos 3\theta + \&c}$

$= 2\left\{\frac{x}{1-x^2}\cos\theta + \frac{x^2}{2(1-x^4)}\cos 2\theta + \frac{x^3}{3(1-x^6)}\cos 3\theta + \&c\right\}$

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Cor 1. $\frac{1}{1^3(e^{2\pi}-1)} + \frac{1}{2^3(e^{4\pi}-1)} + \frac{1}{3^3(e^{6\pi}-1)} + \&c = \frac{7\pi^3}{360} - \frac{S_3}{2}$.

Cor 2. $\frac{1}{1^7(e^{2\pi}-1)} + \frac{1}{2^7(e^{4\pi}-1)} + \frac{1}{3^7(e^{6\pi}-1)} + \&c = \frac{19\pi^7}{113400} - \frac{S_7}{2}$.

16. If $\alpha\beta = \pi$, $f(x) = \int_0^\infty e^{-x^2}\phi(nx)\,dx$ and $f(x)f(x') = f(0)f(0)$, then

$$f(0)\sqrt{\alpha}\left\{\tfrac{1}{2}\phi(0) + e^{-\alpha^2}\phi(n\alpha) + e^{-4\alpha^2}\phi(2n\alpha) + e^{-9\alpha^2}\phi(3n\alpha) + \&c\right\}$$
$$= f(\alpha)\sqrt{\beta}\left\{\tfrac{1}{2}\phi(0) + e^{-\beta^2}\phi(n\beta i) + e^{-4\beta^2}\phi(2n\beta i) + e^{-9\beta^2}\phi(3n\beta i) + \&c\right\}$$

17. $1 + \frac{2\cos nx}{1+x^2} + \frac{2\cos 2nx}{1+4x^2} + \frac{2\cos 3nx}{1+9x^2} + \&c$

$= \frac{\pi}{x}\coth\frac{\pi}{x}\cosh n - \frac{\pi}{x}\sinh n$

if n lies between 0 and $\frac{2\pi}{x}$.

18. $\phi(0) + \frac{\phi(xi)+\phi(-xi)}{1+x^2} + \frac{\phi(2xi)+\phi(-2xi)}{1+4x^2} + \&c$

$= \frac{\pi}{2x}\coth\frac{\pi}{x}\left\{\phi(1)+\phi(-1)\right\} + \frac{\pi}{2x}\left\{\phi(1)-\phi(-1)\right\}$.

N.B. Within certain limits of x only this theorem is true; so we must be very careful in applying this theorem. e.g.

$1 + \frac{2\cos mx\cos nx}{1+x^2} + \frac{2\cos 2mx\cos 2nx}{1+4x^2} + \&c$

$= \frac{\pi}{2x}\coth\frac{\pi}{x}\cosh m\cosh n - \sinh m\cosh n$

is true only when $m+n$ & $m-n$ lies between 0 and $\frac{2\pi}{x}$.

We may write $\phi\{f(x)+\psi(x)\}$ for $\phi(x)$.

This theorem is always true when $x=0$

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$$\log_e \frac{1+4\left(\frac{x}{1-x}-\frac{x^3}{1-x^3}+\frac{x^5}{1-x^5}-\frac{x^7}{1-x^7}+\&c\right)}{1+4\cos 2\theta\left(\frac{x\cos 2\theta}{1-x}-\frac{x^3\cos 3\theta}{1-x^3}+\frac{x^5\cos 5\theta}{1-x^5}-\&c\right)}$$

$$= 4\left\{\frac{x\sin^2\theta}{1(1+x)}-\frac{x^2\sin^2 2\theta}{2(1+x^2)}+\frac{x^3\sin^2 3\theta}{3(1+x^3)}-\&c\right\}$$

$$\frac{1}{4}\log_e \frac{\sin\theta - x\sin 3\theta + x^3\sin 5\theta - x^6\sin 7\theta + \&c}{\sin\theta(1-3x+5x^2-7x^6+9x^{10}-\&c)}$$

$$= 4\left\{\frac{x\sin^2\theta}{1(1-x)}+\frac{x^2\sin^2 2\theta}{2(1-x^2)}+\frac{x^3\sin^2 3\theta}{3(1-x^3)}+\&c\right\}.$$

$$\frac{1}{4}\log_e \frac{\sin 2\theta - x\sin 4\theta + x^5\sin 8\theta - x^8\sin 10\theta + \&}{\sin 2\theta(1-2x+4x^5-5x^8+7x^{16}-\&)}$$

$$= \frac{x}{1+x}\sin^2\theta + \frac{x^2\sin^2 2\theta}{2(1+x^2)} + \frac{x^3\sin^2 3\theta}{3(1+x^3)} + \&c$$

$$+ \frac{x^4}{1-x^4}\sin^2 4\theta + \frac{x^8}{2(1-x^8)}\sin^2 4\theta + \&c$$

$$\frac{1}{4}\log_e \frac{\sin\theta - x\sin 5\theta + x^2\sin 7\theta - x^5\sin 11\theta + \&}{\sin\theta(1-5x+7x^2-11x^5+13x^7-\&)}$$

$$= \frac{x\sin^2\theta}{1-x} + \frac{x^2\sin^2 2\theta}{2(1-x^2)} + \frac{x^3\sin^2 3\theta}{3(1-x^3)} + \&$$

$$+ \frac{x\sin^2 2\theta}{1+x} + \frac{x^2\sin^2 4\theta}{2(1+x^2)} + \frac{x^3\sin^2 6\theta}{3(1+x^3)} + \&c$$

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19 If $\alpha\beta = \pi$, then

$$\left\{\frac{\alpha^2}{6}\phi(0) + \frac{\alpha n}{2}\phi'(0) + \frac{n^2}{4}\phi''(0)\right\}$$

$$+\left\{\frac{\phi(n\alpha)+\phi(-n\alpha)}{1(e^{2\alpha^2}-1)} + \frac{\phi(2n\alpha)+\phi(-2n\alpha)}{2(e^{4\alpha^2}-1)} + \frac{\phi(3n\alpha)+\phi(-3n\alpha)}{3(e^{6\alpha^2}-1)} + \&c\right\}$$

$$+\left\{\phi(n\alpha) + \frac{1}{2}\phi(2n\alpha) + \frac{1}{3}\phi(3n\alpha) + \&c\right\}$$

$$= \left\{\frac{\beta^2}{6}\phi(0) + \frac{\beta n i}{2}\phi'(0) + \frac{\alpha\beta i}{2}\phi(0)\right\}$$

$$+\left\{\frac{\phi(n\beta i)+\phi(-n\beta i)}{1(e^{2\beta^2}-1)} + \frac{\phi(2n\beta i)+\phi(-2n\beta i)}{2(e^{4\beta^2}-1)} + \frac{\phi(3n\beta i)+\phi(-3n\beta i)}{3(e^{6\beta^2}-1)} + \&c\right\}$$

$$+\left\{\phi(n\beta i) + \frac{1}{2}\phi(2n\beta i) + \frac{1}{3}\phi(3n\beta i) + \&c\right\}$$

20. If $\alpha\beta = \pi$, m any even integer and $\psi(m) = \frac{B_m}{\lfloor m}\{(\alpha i)^m + \beta^m\}$

$$+ \frac{B_2}{\lfloor 2}\frac{B_{m-2}}{\lfloor m-2}(\alpha\beta)^2\{(\alpha i)^{m-4} + \beta^{m-4}\} - \frac{B_4}{\lfloor 4}\cdot\frac{B_{m-4}}{\lfloor m-4}(\alpha\beta)^4\{(\alpha i)^{m-8} + \beta^{m-8}\}$$

$+ \&c$ the last term being $-(\alpha\beta i)^{\frac{m}{2}-1}\frac{B_{\frac{m}{2}-1}}{\lfloor \frac{m}{2}-1}\frac{B_{\frac{m}{2}+1}}{\lfloor \frac{m}{2}+1}\{(\alpha i)^2 + \beta^2\}$

or $-(\alpha\beta i)^{\frac{m}{2}}\frac{B_{\frac{m}{2}}}{\lfloor \frac{m}{2}}\cdot\frac{B_{\frac{m}{2}}}{\lfloor \frac{m}{2}}$ according as $\frac{m}{2}$ is odd or even, then

$$\alpha\left\{\frac{\phi(2n\alpha)+\phi(-2n\alpha)}{(2\alpha)^{m-1}(e^{2\alpha^2}-1)} + \frac{\phi(4n\alpha)+\phi(-4n\alpha)}{(4\alpha)^{m-1}(e^{4\alpha^2}-1)} + \&c\right\} + \alpha\left\{\frac{\phi(2n\alpha)}{(2\alpha)^{m-1}} + \frac{\phi(4n\alpha)}{(4\alpha)^{m-1}}\right\}$$

$$= \beta i\left\{\frac{\phi(2n\beta i)+\phi(-2n\beta i)}{(2\beta i)^{m-1}(e^{2\beta^2}-1)} + \frac{\phi(4n\beta i)+\phi(-4n\beta i)}{(4\beta i)^{m-1}(e^{4\beta^2}-1)} + \&c\right\}$$

$$+ \frac{\beta i}{2}\left\{\frac{\phi(2n\beta i)+\phi(-2n\beta i)}{(2\beta i)^{m-1}} + \frac{\phi(4n\beta i)+\phi(-4n\beta i)}{(4\beta i)^{m-1}} + \&c\right\}$$

$$+ \frac{\alpha}{2}\left\{\frac{B_{m-2}}{\lfloor m-2}\beta^{m-2}\frac{n}{\lfloor 2}\phi'(0) + \frac{B_{m-4}}{\lfloor m-4}\beta^{m-4}\frac{n^3}{\lfloor 3}\phi'''(0) + \cdots - \frac{n^{m-1}}{\lfloor m-1}\phi^{m-1}\frac{\phi^m}{2}\right\}$$

$$+ \phi(0)\psi(m) + \frac{n^2}{\lfloor 2}\phi''(0)\psi(m-2) + \cdots\cdots + \frac{n^m}{\lfloor m}\phi^m(0)\psi(0)$$

This theorem is always true whatever be the function ϕ

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$$\frac{\phi'(x)}{\phi(x)} - \frac{\psi'(x)}{\psi(x)} = \frac{1-\phi^4(x)}{8x}$$

Show that

$$\frac{\psi'(x)}{\psi(x)} - 2x\frac{\psi(x^2)}{\psi(x)} = \frac{1-\phi^4(x)}{8x}$$

$$\frac{\phi'(x)}{\phi(x)} \cdot \frac{\phi(-x)}{\phi(x)} = \frac{\phi^4(x) - \phi^4(x)}{4x}$$

$$\frac{1^3 - 3^3x + 5^3x^2 - 7^3x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c}$$

$$= 1 - 24\left\{\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right\}$$

$$\therefore \frac{1^3 - 3^3x^4 + 5^3x^{12} - 7^3x^{24} + \&c}{1 - 3x^4 + 5x^{12} - 7x^{24} + \&c}$$

$$- \frac{1^3 - 3^3x + 5^3x^3 - 7^3x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c} = 3\phi^4(x)$$

$$\frac{1}{240}\cdot\frac{1^5 - 3^5x + 5^5x^3 - 7^5x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c}$$

$$= \frac{1}{240} + \frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \&c$$

$$- 3\left\{\frac{x}{(1-x)^2} + \frac{4x^2}{(1-x^2)^2} + \frac{9x^3}{(1-x^3)^2} + \&c\right\}$$

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1. $\left(\frac{1}{x-n+1} - \frac{1}{x-n+3} + \frac{1}{x-n+5} - \&c\right)$
$-\left(\frac{1}{x+n+1} - \frac{1}{x+n+3} + \frac{1}{x+n+5} - \&c\right)$
$= \frac{n}{x^2-1} + \frac{2^2-n^2}{1+} \frac{2^2}{x^2-1} + \frac{4^2-n^2}{1+} \frac{4^2}{x^2-1} + \&c$

2. $\phi(0) - \frac{1}{2}\phi(1) + \frac{1.3}{2.4}\phi(2) - \&c$
$= \phi(-\frac{1}{2}) - \frac{1}{2}\phi(-1\frac{1}{2}) + \frac{1.3}{2.4}\phi(-2\frac{1}{2}) - \&c$

3. $\frac{1}{4}\phi^2(x) = \frac{1}{4} + \frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \frac{x^7}{1-x^7} + \&c$

4. $\frac{1}{8}\phi^4(x) = \frac{1}{8} + \frac{x}{1-x} + \frac{2x^2}{1+x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1+x^4} + \&c$

5. If $\alpha\beta = \pi^2$, then
$\sqrt{\alpha}\left\{\frac{1}{4} + \frac{1}{e^{\alpha}-1} - \frac{1}{e^{3\alpha}-1} + \frac{1}{e^{5\alpha}-1} - \&c\right\}$
$= \sqrt{\beta}\left\{\frac{1}{4} + \frac{1}{e^{\beta}-1} - \frac{1}{e^{3\beta}-1} + \frac{1}{e^{5\beta}-1} - \&c\right\}$

6. If $\int_0^\infty \phi(x)\cos nx\,dx = \psi(n)$, then
$\int_0^\infty \psi(x)\cos nx\,dx = \frac{\pi}{2}\phi(n)$.

7. If $\int_0^\infty \phi(x)\cos nx\,dx = \psi(n)$, then
(i) $\int_0^\infty \psi(x)\,dx = \frac{\pi}{2}\phi(0)$, (ii) $\int_0^\infty \psi^2(x)\,dx = \frac{\pi}{2}\int_0^\infty \phi^2(x)\,dx$

8. If $\int_r^\infty \phi(x)\cos nx\,dx = \psi(n)$, then
(i) $\int_0^\infty x^n \phi(x)\cos nx \cos\frac{n\pi}{2}\,dx = \psi^{(n)}$
(ii) $\int_0^\infty x^n \psi(x)\cos nx \cos\frac{n\pi}{2}\,dx = \frac{\pi}{2}\phi^{(n)}$.

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$1 - 5x + 7x^2 - 11x^5 + 13x^7 - \&c = \phi^2(-x) f(-x, -x^2) = B_x$

$f(-x, -x^2) f(-x^2, -x^4) = \phi(-x)\psi(x)$

$$\frac{f(-x, -x^4)}{f(x^4, -x^8)} = \frac{\phi(-x^2)}{\psi(x)}$$

$1 - 2x + 4x^5 - 5x^8 + 7x^{16} - \&c = \psi(x^2) \underbrace{f^2(-x, -x^4)}_{= A_x}$

Show that $\frac{1}{4}\phi^2(x)\phi^2(x^4) =$

$$\frac{1}{4} + \frac{x}{1-x} + \frac{x^2}{1+x^2} + \frac{3x^3}{1-x^3} + \frac{6x^4}{1+x^4} + \frac{5x^5}{1-x^5} + \frac{3x^6}{1+x^6} + \&c$$

$$\frac{1}{2}\phi(0) + \phi(1) + \phi(2) + \phi(3) + \&c$$

$$= \int_0^\infty \phi(x) - \frac{\phi(ix) - \phi(-ix)}{i(e^{2\pi x} - 1)}\, dx.$$

$A_x + 2x A_{x^4} = B_{x^8}$.

$1 - 5(x^1) - 7(x^3)^2 + 11(x^3)^5 + 13(x^7)^7 - \&c$

$= \phi^2(x)\psi(-x) + 3x\,\phi^2(x^3).\psi(-x^3).$

$$\frac{(1-x)^5(1-x^2)^5(1-x^3)^5(1-x^4)^5 \&c}{(1-x^5)(1-x^{10})(1-x^{15})(1-x^{20}) \&c}$$

$$= 1 - 5\left(\frac{x}{1+x} - \frac{3x^3}{1+x^3} + \frac{4x^4}{1+x^4} - \frac{7x^7}{1+x^7} + \frac{9x^9}{1+x^9} \right.$$

$$\left. + \frac{11x^{11}}{1+x^{11}} - \frac{12x^{12}}{1+x^{12}}\right)$$

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9. $\phi(x)\,\phi(x^4) = 1 + \frac{2x}{1-x} + \frac{2x^3}{1-x^3} - \frac{2x^5}{1-x^5} - \frac{2x^7}{1-x^7} + \&c$

10. $\phi(x)\,\phi(x^3) = 1 + \frac{2x}{1-x} - \frac{2x^2}{1+x^2} + \frac{2x^5}{1+x^5} - \frac{2x^5}{1-x^5} + \frac{2x^7}{1-x^7} - \&c$

shew that

11. $\phi(x)\,\phi(x^4) = 1 + \frac{2x}{1-x} - \frac{2x^2}{1+x^2} - \frac{2x^3}{1-x^3} + \frac{2x^5}{1-x^5} + \frac{2x^6}{1+x^6} - \frac{2x^7}{1-x^7} + \frac{2x^9}{1-x^9} - \frac{2x^{10}}{1+x^{10}} \&c$

12. $1 - 3x + 5x^3 - 7x^6 + 9x^{10} - \&c$
$= \{(1-x)(1-x^4)(1-x^9)(1-x^{16}) \&c\}^3 = \phi^2(-x)\,\psi(x).$

13. If $\alpha\beta = \frac{\pi}{4}$, then

$\sqrt{\alpha}\{\alpha e^{-\alpha^2} - 3\alpha e^{-9\alpha^2} + 5\alpha e^{-25\alpha^2} - \&c\}$

$= \sqrt{\beta}\{\beta e^{-\beta^2} - 3\beta e^{-9\beta^2} + 5\beta e^{-25\beta^2} - \&c\}$.

14. If $\int_0^\infty \phi(x)\cos 2nx\,dx = \psi(n)$, then

$\alpha\{\frac{1}{2}\phi(0) + \phi(\alpha)\cos n\alpha + \phi(2\alpha)\cos 4n\alpha + \phi(3\alpha)\cos 6n\alpha + \&c\}$

$= \psi(n) + \psi(\beta - n) + \psi(\beta + n) + \psi(2\beta - n) + \psi(2\beta + n) + \&c$

with $\alpha/\beta = \pi$ & n lying between 0 & β.

N. B. $\int_0^\infty \frac{e^{ax} - e^{-ax}}{e^{\pi x} - e^{-\pi x}} \cos mx\,dx = \frac{\sin a}{e^m + 2\cos a + e^{-m}}$

$\int_0^\infty \frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}} \sin mx\,dx = \frac{1}{2}\frac{e^m - e^{-m}}{e^m + 2\cos a + e^{-m}}$

$\int_0^\infty \frac{\sin mx}{e^{2\pi x} - 1}\,dx = \frac{1}{2}\left(\frac{1}{e^m - 1} + \frac{1}{2} - \frac{1}{m}\right)$. $\int_0^\infty \frac{\phi(ax) - \phi(bx)}{x}\,dx$

$= \log\frac{b}{a}$.

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$$y = \pi \cdot \frac{1 + \left(\frac{1}{2}\right)^2 (1-x) + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 (1-x)^2 + \cdots}{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots}$$

$$\text{then } \frac{dy}{dx} = -\frac{1}{x(1-x)\left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}^2}$$

$$\left(\frac{1}{2}\right)^2 + 2\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x + 3\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 x^2 + \cdots$$

$$= \left\{\frac{1}{\pi\sqrt{x(1-x)}} \cdot \frac{1}{12x}\right\}\left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}$$

$$+ \frac{1 - 24\left(\frac{1}{e^{2\pi} - 1} + \frac{2}{e^{4\pi} - 1} + \frac{3}{e^{6\pi} - 1} + \cdots\right)}{12x(1-x)\left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}}$$

$$1 + 240\left(\frac{1^3}{e^{2\pi} - 1} + \frac{2^3}{e^{4\pi} - 1} + \frac{3^3}{e^{6\pi} - 1} + \cdots\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}^4 (1 + 14x + x^2)$$

$$1 - 504\left(\frac{1^5}{e^{2\pi} - 1} + \frac{2^5}{e^{4\pi} - 1} + \frac{3^5}{e^{6\pi} - 1} + \cdots\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}^6 (1 + x)(1 - 34x + x^2)$$

If x is changed to $\left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}}\right)^2$ then

y is changed to $2y$

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$$1 + 240\left(\frac{1^3}{e^{2x}-1} + \frac{2^3}{e^{4x}-1} + \cdots\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}^4 (1 - x + x^2)$$

$$1 - 504\left(\frac{1^5}{e^{2x}-1} + \frac{2^5}{e^{4x}-1} + \cdots\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}^6 (1+x)\left(1 - \frac{x}{2}\right)(1 - 2x) \ldots$$

$$1 - 8\left(\frac{1}{e^x+1} - \frac{2}{e^{2x}+1} + \frac{3}{e^{3x}+1} - \cdots\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}^2 (1 - x)$$

$$1 + 16\left(\frac{1^3}{e^x+1} - \frac{2^3}{e^{2x}+1} + \frac{3^3}{e^{3x}+1} - \cdots\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}^4 (1 - x^2)$$

$$1 - 8\left(\frac{1^5}{e^x+1} - \frac{2^5}{e^{2x}+1} + \frac{3^5}{e^{3x}+1} - \cdots\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}^6 (1 - x)(1 - x + x^2) \ldots$$

$$17 + 32\left(\frac{1^7}{e^x+1} - \frac{2^7}{e^{2x}+1} + \frac{3^7}{e^{3x}+1} - \cdots\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}^8 (1 - x)(17 - 32x + x^2)$$

$$1 + 24\left(\frac{1}{e^x+1} + \frac{2}{e^{2x}+1} + \frac{3}{e^{3x}+1} + \cdots\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \cdots\right\}^2 (1 + x)$$

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$$1 + 4\left(\frac{1}{e^{x}+e^{-x}} + \frac{1}{e^{2x}+e^{-2x}} + \&c\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}$$

$$4\left(\frac{1^2}{e^{x}+e^{-x}} + \frac{2^2}{e^{2x}+e^{-2x}} + \frac{3^2}{e^{3x}+e^{-3x}} + \&c\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}\left(\frac{x}{4}\right)^3$$

$$4\left(\frac{1^4}{e^{x}+e^{-x}} + \frac{2^4}{e^{2x}+e^{-2x}} + \frac{3^4}{e^{3x}+e^{-3x}} + \&c\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}\left\{\left(\frac{x}{2}\right)^5 + \left(\frac{x}{4}\right)^5\right\}$$

$$4\left(\frac{1^6}{e^{x}+e^{-x}} + \frac{2^6}{e^{2x}+e^{-2x}} + \frac{3^6}{e^{3x}+e^{-3x}} + \&c\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}\left\{\left(\frac{x}{2}\right)^7 + 11\left(\frac{x}{4}\right)^7 + \left(\frac{x}{4}\right)^7\right\}$$

$$4\left(\frac{1^8}{e^{x}+e^{-x}} + \frac{2^8}{e^{2x}+e^{-2x}} + \frac{3^8}{e^{3x}+e^{-3x}} + \&c\right)$$

$$= \left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}\left\{\left(\frac{x}{2}\right)^9 + 57\left(\frac{x}{4}\right)^9 + 102\left(\frac{x}{4}\right)^9 + \left(\frac{x}{4}\right)^9\right\}$$

If $\alpha\beta = \pi^2$, then

$$\alpha\left\{\frac{\sec n\alpha}{4} + \frac{\operatorname{cosec} n\alpha}{e^{n\alpha}-1} - \frac{\cos 3n\alpha}{e^{3n\alpha}-1} + \frac{\operatorname{cosec} 3n\alpha}{e^{3n\alpha}-1} - \&c\right\}$$

$$= \beta\left\{\frac{1}{4} + \frac{\operatorname{csch} 2n\beta}{e^{2n\beta}+e^{-2n\beta}} + \frac{\operatorname{csch} 4n\beta}{e^{4n\beta}+e^{-4n\beta}} + \&c\right\}$$

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$$1 + 4\left(\frac{1}{e^{2\pi}-1} - \frac{1}{e^{3\pi}-1} + \frac{1}{e^{5\pi}-1} - \&c\right)$$
$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 1}{2\cdot 4}\right)^2 x^2 + \&c\right\}.$$

$$1 - 4\left(\frac{1^2}{e^{2\pi}-1} - \frac{3^2}{e^{3\pi}-1} + \frac{5^2}{e^{5\pi}-1} - \&c\right)$$
$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \tfrac{1}{2}\left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}^3 (1-x).$$

$$5 + 4\left(\frac{1^4}{e^{2\pi}-1} - \frac{3^4}{e^{3\pi}-1} + \frac{5^4}{e^{5\pi}-1} - \&c\right)$$
$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 1}{2\cdot 4}\right)^2 x^2 + \&c\right\}^5 (5-x)(1-x).$$

$$61 - 4\left(\frac{1^6}{e^{2\pi}-1} - \frac{3^6}{e^{3\pi}-1} + \frac{5^6}{e^{5\pi}-1} - \&c\right)$$
$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 1}{2\cdot 4}\right)^2 x^2 + \&c\right\}^7 (1-x)(61 - 46x + x^2)$$

$$1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \&c\right)$$
$$= \phi^8(-x) + 256\, x\, \psi^8(x).$$

Similarly it is possible to express any identity in terms of $\phi(x)$ & $\psi(x)$.

$$\int \frac{\phi^{2n+4}(u)}{u}\, du = \int \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}^n \frac{dx}{x(1-x)}$$

where $x = 1 - \dfrac{\phi^4(u)}{\phi^4(u)}$.

or $u = F(x)$ or $e^{-\pi}$.

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$$1 - 16\left(\frac{1^3}{e^{2y}-1} - \frac{2^3}{e^{4y}-1} + \frac{3^3}{e^{6y}-1} - \&c\right)$$
$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{1\cdot 2}\right)^2 x^2 + \&c\right\}^4 (1-x)^2.$$

$$1 + 8\left(\frac{1^5}{e^{2y}-1} - \frac{2^5}{e^{4y}-1} + \frac{3^5}{e^{6y}-1} - \&c\right)$$
$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{1\cdot 2}\right)^2 x^2 + \&c\right\}^6 (1-x)(1-x^2).$$

$$17 - 32\left(\frac{1^7}{e^{2y}-1} - \frac{2^7}{e^{4y}-1} + \frac{3^7}{e^{6y}-1} - \&c\right)$$
$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{1\cdot 2}\right)^2 x^2 + \&c\right\}^8 (1-x)^2(17 - 2x + 17x^2)$$

$$31 + 8\left(\frac{1^9}{e^{2y}-1} - \frac{2^9}{e^{4y}-1} + \frac{3^9}{e^{6y}-1} - \&c\right)$$
$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{1\cdot 2}\right)^2 x^2 + \&c\right\}^{10} (1-x)(1-x^2)(31 - 46x + 31x^2).$$

$$\frac{1^3}{e^{2y} - e^{-2y}} + \frac{2^3}{e^{4y} - e^{-4y}} + \frac{3^3}{e^{6y} - e^{-6y}} + \&c$$
$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{1\cdot 2}\right)^2 x^2 + \&c\right\}^4 \frac{x}{16}.$$

$$\frac{1^5}{e^{2y} - e^{-2y}} + \frac{2^5}{e^{4y} - e^{-4y}} + \frac{3^5}{e^{6y} - e^{-6y}} + \&c$$
$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{1\cdot 2}\right)^2 x^2 + \&c\right\}^6 \frac{x(1+x)}{16}$$

$$\frac{1^7}{e^{2y} - e^{-2y}} + \frac{2^7}{e^{4y} - e^{-4y}} + \frac{3^7}{e^{6y} - e^{-6y}} + \&c$$
$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{1\cdot 2}\right)^2 x^2 + \&c\right\}^8 \frac{x(1 + 6\frac{1}{2}x + x^2)}{16}.$$

$$\frac{1^9}{e^{2y} - e^{-2y}} + \frac{2^9}{e^{4y} - e^{-4y}} + \frac{3^9}{e^{6y} - e^{-6y}} + \&c$$
$$= \left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{1\cdot 2}\right)^2 x^2 + \&c\right\}^{10} \frac{x(1+x)(1 + 29x + x^2)}{16}.$$

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$$1 + 24\left(\frac{1}{e^{2x}-1} + \frac{2}{e^{2x}-1} + \frac{3}{e^{2x}-1} + \cdots\right)$$

$$= \{1 + (1)x + (1)x^2 + \cdots\} \int^4 (1+x).$$

$$1 - 240\left(\frac{1^3}{e^{2x}-1} + \frac{2^3}{e^{2x}-1} + \frac{3^3}{e^{2x}-1} + \cdots\right)$$

$$= \{1 + (1)x + (1)x^2 + \cdots\} \int^4 (1 - 16x + x^4).$$

$$1 + 504\left(\frac{1^5}{e^{2x}-1} + \frac{2^5}{e^{2x}-1} + \frac{3^5}{e^{2x}-1} + \cdots\right)$$

$$= z^6(1+x)(1+29x+x^2).$$

$$1 + 24\left(\frac{1}{e^{2x}-1} + \frac{2}{e^{2x}-1} + \frac{3}{e^{2x}-1} + \cdots\right)$$

$$= z^2\left(1 - \frac{x}{2}\right).$$

$$1 - 16\left(\frac{1^3}{e^{2x}-1} - \frac{2^3}{e^{2x}-1} + \frac{3^3}{e^{2x}-1} - \cdots\right)$$

$$= z^4(1-x).$$

$$1 + 8\left(\frac{1^5}{e^{2x}-1} - \frac{2^5}{e^{2x}-1} + \frac{3^5}{e^{2x}-1} - \cdots\right)$$

$$= z^6(1-x)\left(1 - \frac{x}{2}\right).$$

$$17 - 32\left(\frac{1^7}{e^{2x}-1} - \frac{2^7}{e^{2x}-1} + \frac{3^7}{e^{2x}-1} - \cdots\right)$$

$$= z^8(1-x)(17 - 17x + 2x^2)$$

$$1 + 240\left(\frac{1^3}{e^{2x}-1} + \frac{2^3}{e^{2x}-1} + \frac{3^3}{e^{2x}-1} + \cdots\right) = z^4\left(1 - x + \frac{x^2}{16}\right)$$

$$1 - 504\left(\frac{1^5}{e^{2x}-1} + \frac{2^5}{e^{2x}-1} + \frac{3^5}{e^{2x}-1} + \cdots\right) = z^6\left(1 - \frac{x}{2}\right)(1 - x + \frac{x^2}{32})$$

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$\alpha\beta = \pi^2$ &c &c

$$\frac{\sqrt{\alpha}}{\alpha^n}\left\{\frac{1}{2\alpha}\left(\frac{1}{1^{2n}}-\frac{1}{3^{2n}}+\frac{1}{5^{2n}}-\&c\right)+\frac{1}{1^{2n}(e^{\alpha}-1)}-\frac{1}{3^{2n}(e^{3\alpha}-1)}\right.$$
$$\left.+\frac{1}{5^{2n}(e^{5\alpha}-1)}-\frac{1}{7^{2n}(e^{7\alpha}-1)}+\&c\right\}$$
$$=\frac{\sqrt{\beta}}{\beta^n}\left\{(-1)^n\left[\frac{1}{2^{2n}(e^{\beta}+e^{-\beta})}+\frac{1}{1^{2n}(e^{4\beta}+e^{-4\beta})}+\&c\right]\right.$$
$$+\frac{1}{4}\left[\frac{\left(\frac{\beta}{2}\right)^{2n}}{\lfloor 2n}E_{2n+1}+\frac{B_2}{\lfloor 2}\cdot\frac{E_{2n-1}}{\lfloor 2n-2}\cdot\left(\frac{\beta}{2}\right)^{2n-1}(2\alpha)\right.$$
$$\left.-\frac{B_4}{\lfloor 4}\cdot\frac{E_{2n-3}}{\lfloor 2n-4}\left(\frac{\beta}{2}\right)^{2n-4}(2\alpha)^2+\&c\ \&c\right.$$
$$\left.\left.-\frac{(-\alpha\beta)^n}{\lfloor 2n}\left[B_{2n}E_{1^n}\right]\right]\right\}$$

$$\log\frac{1+\frac{4x}{1+x^2}+\frac{4x^2}{1+x^4}+\frac{4x^3}{1+x^6}+\&c}{1+\frac{4x\cos 2\alpha}{1+x^2}+\frac{4x^2\cos 4\alpha}{1+x^4}+\&c}$$
$$=8\left\{\frac{x\sin^2\alpha}{1(1-x^2)}+\frac{x^3\sin^2 3\alpha}{3(1-x^6)}+\&c\right\}.$$
$$1+\frac{4x\cos\alpha}{1+x^2}+\frac{4x^2\cos 2\alpha}{1+x^4}+\frac{4x^3\cos 3\alpha}{1+x^6}+\&c$$
$$=\frac{(1+x)^3(1-2x^2\cos 2\alpha+2x^8\cos 4\alpha-2x^{18}\cos 6\alpha+\&c}{(1-2x\cos\alpha+2x^4\cos 2\alpha-2x^9\cos 3\alpha+\&c)^2}$$

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$$1 + \frac{4x\cos n}{1+x^2} + \frac{4x^4\cos 2n}{1+x^4} + \ldots$$

$$\phi^2(-x^2) \cdot \frac{1 + 2x\cos n + 2x^4\cos 2n + 2x^9\cos 3n + \ldots}{1 - 2x\cos n + 2x^4\cos 2n - 2x^9\cos 3n + \ldots}$$

$$\frac{1}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}} - \frac{3}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}} + \frac{5}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}} - \&c$$

$$= \frac{z^2}{4}\sqrt{x(1-x)}.$$

$$\frac{1^3}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}} - \frac{3^3}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}} + \frac{5^3}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}} - \&c$$

$$= \frac{z^4}{4}\sqrt{x(1-x)}\,(1-2x).$$

$$\frac{1^5}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}} - \frac{3^5}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}} + \frac{5^5}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}} - \&c$$

$$= \frac{z^6}{4}\sqrt{x(1-x)}\,(1-16x+16x^2).$$

$$\frac{1^7}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}} - \frac{3^7}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}} + \frac{5^7}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}} - \&c$$

$$= \frac{z^8}{4}\sqrt{x(1-x)}\,(1-2x)(1-136x+136x^2).$$

$$\frac{1^9}{e^{\frac{y}{2}}+e^{-\frac{y}{2}}} - \frac{3^9}{e^{\frac{3y}{2}}+e^{-\frac{3y}{2}}} + \frac{5^9}{e^{\frac{5y}{2}}+e^{-\frac{5y}{2}}} - \&c$$

$$= \frac{z^{10}}{4}\sqrt{x(1-x)}\,\{1-1232x(1-x)+7936x^2(1-x)^2\}$$

The last term $= 2^{2n}(2^n-1)\frac{B_n}{n}$.

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If α/β is odd integer then

$$\therefore \alpha^{2n+1}\left\{\frac{1^{2n+1}}{e^{\frac{\alpha^2}{2}}+e^{-\frac{\alpha^2}{2}}} - \frac{3^{2n+1}}{e^{\frac{3\alpha^2}{2}}+e^{-\frac{3\alpha^2}{2}}} + \frac{5^{2n+1}}{e^{\frac{5\alpha^2}{2}}+e^{-\frac{5\alpha^2}{2}}} - \cdots\right.$$

$$\mp(\pm\beta)^{2n+1}\left\{\frac{1^{2n+1}}{e^{\frac{\beta^2}{2}}+e^{-\frac{\beta^2}{2}}} - \frac{3^{2n+1}}{e^{\frac{3\beta^2}{2}}+e^{-\frac{3\beta^2}{2}}} + \frac{5^{2n+1}}{e^{\frac{5\beta^2}{2}}+e^{-\frac{5\beta^2}{2}}} - \cdots\right.$$

$$= 0$$

If $\alpha/\beta = \pi$, then

$$\alpha\left\{\frac{\phi(\alpha)-\phi(-\alpha)}{e^{\frac{\alpha^2}{2}}+e^{-\frac{\alpha^2}{2}}} - \frac{\phi(3\alpha)-\phi(-3\alpha)}{e^{\frac{3\alpha^2}{2}}+e^{-\frac{3\alpha^2}{2}}} + \cdots\right\}$$

$$\mp\beta\left\{\frac{\phi(\beta)-\phi(-\beta)}{e^{\frac{\beta^2}{2}}+e^{-\frac{\beta^2}{2}}} - \frac{\phi(3\beta)-\phi(-3\beta)}{e^{\frac{3\beta^2}{2}}+e^{-\frac{3\beta^2}{2}}} + \cdots\right\}$$

$$= 0.$$

If $\alpha/\beta = \pi$, then

$$\alpha\left\{\frac{\sin n\alpha}{e^{\frac{\alpha^2}{2}}+e^{-\frac{\alpha^2}{2}}} - \frac{\sin 3n\alpha}{e^{\frac{3\alpha^2}{2}}+e^{-\frac{3\alpha^2}{2}}} + \cdots\right\}$$

$$-\beta\left\{\frac{\sinh n\beta}{e^{\frac{\beta^2}{2}}+e^{-\frac{\beta^2}{2}}} - \frac{\sinh 3n\beta}{e^{\frac{3\beta^2}{2}}+e^{-\frac{3\beta^2}{2}}} + \cdots\right\}$$

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If $\alpha\beta = \pi^2$ then

$$\frac{1}{1(e^{\frac{\alpha}{2}}+e^{-\frac{\alpha}{2}})} - \frac{1}{3(e^{\frac{3\alpha}{2}}+e^{-\frac{3\alpha}{2}})} + \frac{1}{5(e^{\frac{5\alpha}{2}}+e^{-\frac{5\alpha}{2}})} - \&c$$

$$+ \frac{1}{1(e^{\frac{\beta}{2}}+e^{-\frac{\beta}{2}})} - \frac{1}{3(e^{\frac{3\beta}{2}}+e^{-\frac{3\beta}{2}})} + \frac{1}{5(e^{\frac{5\beta}{2}}+e^{-\frac{5\beta}{2}})} - \&c$$

$$= \frac{\pi}{8}.$$

If $\alpha\beta = \frac{\pi^2}{4}$, then

$$(\tan^{-1}e^{-\alpha} - \tan^{-1}e^{-3\alpha} + \tan^{-1}e^{-5\alpha} - \&c)$$

$$+(\tan^{-1}e^{-\beta} - \tan^{-1}e^{-3\beta} + \tan^{-1}e^{-5\beta} - \&c) = \frac{\pi}{8}.$$

$$\tan^{-1}e^{-\frac{\gamma}{2}} - \tan^{-1}e^{-\frac{3\gamma}{2}} + \tan^{-1}e^{-\frac{5\gamma}{2}} - \&c$$

$$= \frac{1}{4}\sin^{-1}\sqrt{x}$$

$$\log_e(1-e^{-2\gamma}) + \log_e(1-e^{-4\gamma}) + \log_e(1-e^{-6\gamma}) +$$

$$= \frac{1}{12}\left\{\gamma + \log_e\frac{x(1-x)}{16}\right\} + \frac{1}{2}\log_e\left\{1+\left(\frac{1}{2}\right)^2 x + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \right\}$$

$$-1 - 2A_1\left(\frac{1}{e^{2\gamma}-1} + \frac{2}{e^{4\gamma}-1} + \&c\right)$$

$$= (1-2x)z^2 + 6x(1-x)z\cdot\frac{dz}{dx}.$$

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145 $\frac{1}{e^y - e^{-y}} + \frac{3}{e^{3y} - e^{-3y}} + \frac{5}{e^{5y} - e^{-5y}} + \&c = \frac{x}{16} x^2$.

$= \frac{1^3}{e^y - e^{-y}} + \frac{3^3}{e^{3y} - e^{-3y}} + \frac{5^3}{e^{5y} - e^{-5y}} + \&c$

$= \frac{x}{16} x^4 \left(1 - \frac{x}{2}\right)$

$\frac{1^5}{e^y - e^{-y}} + \frac{3^5}{e^{3y} - e^{-3y}} + \frac{5^5}{e^{5y} - e^{-5y}} + \&c$

$= \frac{x}{16} x^6 (1 - x + x^4)$

$\frac{1^7}{e^y - e^{-y}} + \frac{3^7}{e^{3y} - e^{-3y}} + \frac{5^7}{e^{5y} - e^{-5y}} + \&c$

$= \frac{x}{16} x^8 \left(1 - \frac{x}{2}\right)\left(1 - x + \frac{17}{2} x^4\right)$

$\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c$

$= x \frac{1+x}{(1-x)^2} - x^4 \frac{1+x^2}{(1-x^2)^2} + x^6 \frac{1+x^3}{(1-x^3)^2} - \&c$

$\frac{1+x}{1-x} = x \frac{1+x^3}{1-x^3} + x^3 \frac{1+x^5}{1-x^5} - x^6 \frac{1+x}{1-x^7} + \&c$

$= \psi(x)\, \phi(x)$

$\frac{1+x}{1-x} = x \frac{1+x^3}{1-x^3} + x^6 \frac{1+x^5}{1-x^5} - x^{12} \frac{1+x^7}{1-x^7} + \&c$

$= \psi^2(x)$.

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If $\alpha\beta = \pi^2$, then

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$$\frac{\pi^2}{12} + \frac{1}{1^2(e^{\alpha}-e^{-\alpha})^2} + \frac{1}{2^2(e^{2\alpha}-e^{-2\alpha})^2} + \frac{1}{3^2(e^{3\alpha}-e^{-3\alpha})^2} + \&c$$

$$+ \frac{1}{1^2(e^{\beta}-e^{-\beta})^2} + \frac{1}{2^2(e^{2\beta}-e^{-2\beta})^2} + \frac{1}{3^2(e^{3\beta}-e^{-3\beta})^2} + \&c$$

$$- 2\alpha\left\{1^2\log_e(1-e^{-2\alpha}) + 2^2\log_e(1-e^{-4\alpha}) + 3^2\log_e(1-e^{-6\alpha}) + \&c\right\}$$

$$- 2\beta\left\{1^2\log_e(1-e^{-2\beta}) + 2^2\log_e(1-e^{-4\beta}) + 3^2\log_e(1-e^{-6\beta}) + \&c\right\}$$

$$= \frac{\alpha^2+\beta^2}{120}.$$

If $\alpha\beta = \pi^2$ and n any integer, then

$$\alpha^{1-n}\left\{\frac{1}{1^{2n-1}(e^{\frac{\alpha}{2}}+e^{-\frac{\alpha}{2}})} - \frac{1}{3^{2n-1}(e^{\frac{3\alpha}{2}}+e^{-\frac{3\alpha}{2}})} + \&c\right\}$$

$$+(-\beta)^{1-n}\left\{\frac{1}{1^{2n-1}(e^{\frac{\beta}{2}}+e^{-\frac{\beta}{2}})} - \frac{1}{3^{2n-1}(e^{\frac{3\beta}{2}}+e^{-\frac{3\beta}{2}})} + \&c\right\} =$$

$$= \frac{\pi}{2^{2n+1}}\left[\frac{E_1 E_{2n-1}}{\lfloor 2n-2}\left\{(-\alpha)^{n-1}+\beta^{n-1}\right\} - \frac{E_3 E_{2n-3}}{\lfloor 2 \lfloor 2n-4} \times \right.$$

$$\left.\left\{(-\alpha)^{n-3}+\beta^{n-3}\right\} + \frac{E_5 E_{2n-5}}{\lfloor 4 \lfloor 2n-6}\left\{(-\alpha)^{n-5}+\beta^{n-5}\right\}\right.$$

$-$ &c the last term being $(-1)^{\frac{n-1}{2}}\left(\frac{E_n}{\lfloor n-1}\right)^2$ or

$(-1)^{\frac{n}{2}}\frac{E_{n-1}E_{n+1}}{\lfloor n-2 \lfloor n}(\alpha-\beta)$ according as n is odd or ev

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14. $\frac{1}{1^2(e^{\frac{\pi}{2}}+e^{-\frac{\pi}{2}})} + \frac{1}{3^2(e^{\frac{3\pi}{2}}+e^{-\frac{3\pi}{2}})} + \frac{1}{5^2(e^{\frac{5\pi}{2}}+e^{-\frac{5\pi}{2}})} + \&c$

$= \frac{\sqrt{x}}{4} \frac{1 + (\frac{2}{3})^2 x + (\frac{2\cdot4}{3\cdot5})^2 x^2 + (\frac{2\cdot4\cdot6}{3\cdot5\cdot7})^2 x^3 + \&c}{1 + (\frac{1}{2})^2 x + (\frac{1\cdot3}{2\cdot4})^2 x^2 + (\frac{1\cdot3\cdot5}{2\cdot4\cdot6})^2 x^3 + \&c}$

$\frac{1}{1^2(e^{\frac{\pi}{2}}-e^{-\frac{\pi}{2}})} - \frac{1}{3^2(e^{\frac{3\pi}{2}}-e^{-\frac{3\pi}{2}})} + \frac{1}{5^2(e^{\frac{5\pi}{2}}-e^{-\frac{5\pi}{2}})} - \&c$

$= \frac{\sqrt{x}}{4} \cdot \frac{1 + (\frac{2}{3})^2\{1+(\frac{1}{2})^2\} x + (\frac{2\cdot4}{3\cdot5})^2\{1+(\frac{1}{2})^2+(\frac{1\cdot3}{2\cdot4})^2\} x^2 + \&c}{1 + (\frac{1}{2})^2 x + (\frac{1\cdot3}{2\cdot4})^2 x^2 + (\frac{1\cdot3\cdot5}{2\cdot4\cdot6})^2 x^3 + \&c}$

$\frac{1}{1^4(e^{\frac{\pi}{2}}+e^{-\frac{\pi}{2}})} + \frac{1}{3^4(e^{\frac{3\pi}{2}}+e^{-\frac{3\pi}{2}})} + \frac{1}{5^4(e^{\frac{5\pi}{2}}+e^{-\frac{5\pi}{2}})} + \&c$

$= \frac{\sqrt{x}}{4} \cdot \frac{1 + (\frac{2}{3})^2[1+(\frac{2}{3})^2\{1+(\frac{1}{2})^2\}] x + \&c}{\{1 + (\frac{1}{2})^2 x + (\frac{1\cdot3}{2\cdot4})^2 x^2 + \&c\}^3}$

$\sqrt{1-x} + (\frac{2}{3})^2 \sqrt{1-x}^3 + (\frac{2\cdot4}{3\cdot5})^2 \sqrt{1-x}^5 +$

$= \frac{\pi^2}{4}\{1 + (\frac{1}{2})^2(1-x) + (\frac{1\cdot3}{2\cdot4})^2(1-x)^2 + \}$

$- 2(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c)\{1 + (\frac{1}{2})^2 x + (\frac{1\cdot3}{2\cdot4})^2 x^2 + \}$

$+ 4\{1 + (\frac{1}{2})^2 x + (\frac{1\cdot3}{2\cdot4})^2 x^2 + \}\{\frac{1}{1^2(e^{\pi}+1)} - \frac{1}{3^2(e^{3\pi}+1)}$

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$$\frac{1}{1(e^{\frac{x}{2}}-e^{-\frac{x}{2}})} + \frac{1}{3(e^{\frac{3x}{2}}-e^{-\frac{3x}{2}})} + \frac{1}{5(e^{\frac{5x}{2}}-e^{-\frac{5x}{2}})} \cdots$$

$$= \frac{\sqrt{x}}{2}\left(1 + \frac{x}{3} + \frac{x^2}{5} + \frac{x^3}{7} + \cdots\right)$$

$$\frac{1}{1(e^{\frac{x}{2}}+e^{-\frac{x}{2}})} - \frac{1}{3(e^{\frac{3x}{2}}+e^{-\frac{3x}{2}})} + \frac{1}{5(e^{\frac{5x}{2}}+e^{-\frac{5x}{2}})} \cdots$$

$$= \frac{\sqrt{x}}{2}\left\{1 + \frac{1}{2}\cdot\frac{x}{3} + \frac{1.3}{2.4}\cdot\frac{x^2}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^3}{7} + \cdots\right\}.$$

If $P_1 e^{-x} + P_2 e^{-2x} + P_3 e^{-3x} + \cdots$

$= Q_1 x + Q_2 x^2 + Q_3 x^3 + \cdots = f(x)$.

then $P_1 e^{-x} - P_2 e^{-2x} + P_3 e^{-3x} - \cdots$

$= Q_1\left(\frac{x}{1-x}\right) - Q_2\left(\frac{x}{1-x}\right)^2 + Q_3\left(\frac{x}{1-x}\right)^3 - \cdots = -f\left(-\frac{x}{1-x}\right)$.

If $F(e^{-x}) = f(x)$ then $F(-e^{-x}) = f\left(-\frac{x}{1-x}\right)$.

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$$1 - \frac{4x}{1+x} + \frac{4x^3}{1+x^2} - \frac{4x^6}{1+x^3} + \frac{4x^{10}}{1+x^4} - \&c$$

$$= (1 - 2x + 2x^4 - 2x^9 + 2x^{16} - \&c)^2$$

$$\frac{\psi^3(x)}{\psi(x^3)} = 1 + 3\left(\frac{x}{1-x} - \frac{x^5}{1-x^5} + \frac{x^7}{1-x^7} - \frac{x^{11}}{1-x^{11}} + \&c\right)$$

$$\frac{\phi^3(x)}{\phi(x^3)} = 1 + 6\left(\frac{x}{1-x} + \frac{x^2}{1+x^2} - \frac{x^4}{1+x^4} - \frac{x^5}{1-x^5} + \&c\right).$$

$$\left.\begin{aligned} \frac{\phi^3(x)}{\phi(x^3)} + 2\,\frac{\phi^3(-x^2)}{\phi(-x^6)} &= 3\,\phi(x)\,\phi(x^3). \\ \frac{\phi^3(x)}{\phi(x^3)} + \frac{\phi^3(-x^2)}{\phi(-x^6)} &= 2\,\frac{\psi^3(x)}{\psi(x^3)}. \end{aligned}\right\}$$

$$F\left(\frac{2-\sqrt{3}}{4}\right) = e^{-\pi\sqrt{3}}$$

3. $$\frac{1 + \left(\frac{1}{2}\right)^2(1-\alpha) + \left(\frac{1\cdot3}{2\cdot4}\right)^2(1-\alpha)^2 + \&c}{1 + \left(\frac{1}{2}\right)^2\alpha + \left(\frac{1\cdot3}{2\cdot4}\right)^2\alpha^2 + \&c} = \frac{1 + \left(\frac{1}{2}\right)^2(1-\beta) + \&c}{1 + \left(\frac{1}{2}\right)^2\beta + \&c}$$

then $$\sqrt[8]{\frac{\alpha^3}{\beta}} - \sqrt[8]{\frac{(1-\alpha)^3}{1-\beta}} = 1.$$

and $$\sqrt{\sqrt[8]{\frac{\alpha^3}{\beta}} - \alpha}\left\{1 + \left(\frac{1}{2}\right)^2\alpha + \left(\frac{1\cdot3}{2\cdot4}\right)^2\alpha^2 + \&c\right\}$$

$$= \sqrt{\sqrt[8]{\frac{\alpha^3}{\beta}} - \beta}\left\{1 + \left(\frac{1}{2}\right)^2\beta + \left(\frac{1\cdot3}{2\cdot4}\right)^2\beta^2 + \&c\right\}$$

and $$\sqrt[8]{\frac{\alpha^3}{\beta}} + \sqrt[8]{\frac{(1-\alpha)^3}{1-\beta}} = 3\sqrt{\frac{\sqrt[8]{\frac{\alpha^3}{\beta}} - \alpha}{\sqrt[8]{\frac{\alpha^3}{\beta}} - \beta}}$$

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If $\sin^3\theta = m^4 \sin\phi$ & $\cos^3\theta = n^4 \cos\phi$ 150.
and $m - n = 1$

then $\sqrt{\frac{m - \sin^2\theta}{m - \sin^2\phi}} = \frac{m+n}{3}$.

$$= \frac{1 + (\frac{1}{2})^2 \sin^2\phi + (\frac{1\cdot3}{2\cdot2})^2 \sin^4\phi + \dots}{1 + (\frac{1}{2})^2 \sin^2\theta + (\frac{1\cdot3}{2\cdot4})^2 \sin^4\theta + \dots}$$

If $\frac{\alpha^3}{\beta} = (1+\beta)^8$ & $\frac{(1-\alpha)^3}{1-\beta} = \beta^8$

then $\alpha = \frac{(1+\beta)^2(1-\beta^2)}{1+2\beta}$

& $\beta = \frac{(1-\beta)^2(1-\beta^2)}{(1+2\beta)^3}$.

$$1 - \alpha = \beta^3 \cdot \frac{\beta+2}{1+2\beta}.$$

$$1 - \beta = \beta \cdot \left(\frac{\beta+2}{1+2\beta}\right)^3.$$

$$\left\{1 + (\tfrac{1}{2})^2 \beta \left(\frac{2+\beta}{1+2\beta}\right)^3 + (\tfrac{1\cdot3}{2\cdot4})^2 \beta^2 \left(\frac{2+\beta}{1+2\beta}\right)^6 + \dots\right\}$$
$$= (1+2\beta)\left\{1 + (\tfrac{1}{2})^2 \beta^3 \cdot \frac{2+\beta}{1+2\beta} + (\tfrac{1\cdot3}{2\cdot4})^2 \beta^6 \left(\frac{2+\beta}{1+2\beta}\right)^2 + \dots\right\}$$

If $\beta^3 \cdot \frac{2+\beta}{1+2\beta} = \frac{1-\sqrt{1-x^3}}{2}$.

then $\beta = \frac{-(1+\sqrt{1-x}) + \sqrt{2+x+2\sqrt{1+x}}}{2}$

If $\sqrt[8]{\alpha} = m$ & $\sqrt[8]{\beta} = n$

$$m^4 + 2m^3n^3 - 2mn - n^4 = 0.$$

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(5) $F\left\{\frac{1}{2}-(2-\sqrt{3})\sqrt[4]{12}\right\} = e^{-3\pi}$

$\phi(e^{-3\pi}) = \frac{\phi(e^{-\pi})}{\sqrt[4]{6\sqrt{3}-9}}$

$1+\frac{2}{\pi}\cdot\frac{m}{2m}\cdot x\cdot\left(\frac{2+x}{1+2x}\right)^3 + \&c$

$=(1+2x)^{2n}\left\{1+\frac{1}{2}\cdot\frac{2}{\pi}\,x^3\,\frac{2+x}{1+2x}+\&c\right\}$ if $m=\frac{4n+1}{6}$.

$x\,\frac{\psi^3(x^2)}{\psi(x)} = \frac{x}{1-x^2}-\frac{x^2}{1-x^4}+\frac{x^4}{1-x^8}-\frac{x^5}{1-x^{10}}+\frac{x^7}{1-x^{12}}$
$-\frac{x^8}{1-x^{16}}+\frac{x^{10}}{1-x^{20}}-\&c$

$\frac{\phi^3(x^3)}{\phi(x)} = 1-2\left(\frac{x}{1+x}-\frac{x^2}{1-x^2}+\frac{x^4}{1-x^4}-\frac{x^5}{1+x^5}+\frac{x^7}{1+x^7}\right.$
$\left.-\frac{x^8}{1-x^8}+\frac{x^{10}}{1-x^{10}}-\frac{x^{11}}{1+x^{11}}+\&c\right)$.

$\frac{\psi^3(-x^3)}{\psi(-x)}-\frac{\psi^3(x^3)}{\psi(x)} = 2x\,\frac{\psi^3(x^6)}{\psi(x^2)}$.

$\frac{\psi^5(-x^5)}{\psi(-x)}-\frac{\psi^5(x^5)}{\psi(x)} = 4x^3\,\frac{\psi^5(x^{10})}{\psi(x^4)}$
$+2x\,\frac{f^5(-x^{10},-x^{30})}{f(-x^4,-x^8)}$.

$8\cdot\frac{(1-\alpha)^3}{\alpha}-\sqrt{\frac{\alpha^3}{\alpha}}=1$

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$$\frac{\frac{1}{2} + e^{-\pi x}\cos(\pi\sqrt{1-x^2}) + e^{-4\pi x}\cos(4\pi\sqrt{1-x^2}) + \text{etc}}{e^{-\pi x}\sin(\pi\sqrt{1-x^2}) + e^{-4\pi x}\sin(4\pi\sqrt{1-x^2}) + \text{etc}} = \frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{1-x}}$$

$$(\sqrt{5}+\sqrt{3})\left\{1 + 2e^{-\frac{\pi\sqrt{5}}{3}} + 2e^{-\frac{4\pi\sqrt{5}}{3}} + 2e^{-\frac{9\pi\sqrt{5}}{3}} + \text{etc}\right\}$$
$$= (3+\sqrt{3})\left\{1 + 2e^{-3\pi\sqrt{5}} + 2e^{-12\pi\sqrt{5}} + 2e^{-27\sqrt{5}}\right\}$$

$$\frac{1}{\phi^4(e^{-\pi})} = .71777$$

$$\phi(e^{-5\pi}) = \frac{\phi(e^{-\pi})}{\sqrt{5-\sqrt{5}-10}}$$

$$\frac{\psi^3(x)}{\psi(x^3)} + \frac{\psi^3(-x)}{\psi(-x^3)} = 2\cdot\frac{\psi^3(x^2)}{\psi(x^6)}$$

$$\frac{\psi^5(x)}{\psi(x^5)} + \frac{\psi^5(-x)}{\psi(-x^5)} + 2\frac{f(-x^4,-x^5)}{f(-x^{20},-x^{40})} = 4\frac{\psi^5(x^2)}{\psi(x^{10})}$$

$$\sqrt[8]{\frac{(1-\beta)^5}{1-\alpha}} - \sqrt[8]{\frac{\beta^5}{\alpha}} = 1 + \frac{3}{2}\cdot\sqrt[24]{\frac{\alpha^4(1-\alpha)^4}{\alpha(1-\alpha)}}$$

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If 5. $\dfrac{1+\left(\frac{1}{2}\right)^2(1-\alpha)+\left(\frac{1\cdot3}{2\cdot4}\right)^2(1-\alpha)^2+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\left(\frac{1\cdot3}{2\cdot4}\right)^2\alpha^2+\&c}$

$= \dfrac{1+\left(\frac{1}{2}\right)^2(1-\beta)+\left(\frac{1\cdot3}{2\cdot4}\right)^2(1-\beta)^2+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\left(\frac{1\cdot3}{2\cdot4}\right)^2\beta^2+\&c}$

then $\sqrt[8]{\dfrac{\alpha^5}{\beta}}-\sqrt[8]{\dfrac{(1-\alpha)^5}{1-\beta}}=1+\sqrt[3]{2}\cdot\sqrt[24]{\dfrac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}}$

If 9. $\left.\begin{array}{l}\dfrac{1+\left(\frac{1}{2}\right)^2(1-\alpha)+\left(\frac{1\cdot3}{2\cdot4}\right)^2(1-\alpha)^2+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\left(\frac{1\cdot3}{2\cdot4}\right)^2\alpha^2+\&c}\\ \dfrac{1+\left(\frac{1}{2}\right)^2(1-\beta)+\left(\frac{1\cdot3}{2\cdot4}\right)^2(1-\beta)^2+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\left(\frac{1\cdot3}{2\cdot4}\right)^2\beta^2+\&c}\end{array}\right\}=3\cdot\dfrac{1+\left(\frac{1}{2}\right)^2(1-\gamma)+}{1+\left(\frac{1}{2}\right)^2\gamma+}$

$\sqrt[8]{\frac{\alpha}{\beta}}\sqrt[4]{\frac{1-\alpha}{1-\beta}}\left\{1+\left(\frac{1}{2}\right)^2\alpha+\left(\frac{1\cdot3}{2\cdot4}\right)^2\alpha^2+\&c\right\}$

$+3\left\{1+\left(\frac{1}{2}\right)^2\beta+\left(\frac{1\cdot3}{2\cdot4}\right)^2\beta^2+\&c\right\}$

$=\left(\sqrt[8]{\frac{\alpha}{\beta}}+\sqrt[4]{\frac{1-\alpha}{1-\beta}}\right)\sqrt{\left\{1+\left(\frac{1}{2}\right)^2\alpha+\&c\right\}\left\{1+\left(\frac{1}{2}\right)^2\beta+\&c\right\}}$

$\dfrac{\phi^2(x)}{\phi(-x^5)}+4\dfrac{(1-x)^5(1-x^4)^5(1-x^6)^5}{(1-x^5)(1-x^{10})(1-x^{15})}=$

$=5\,\phi^2(x)\,\phi(-x^5)$

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$$1+2\sqrt[8]{\frac{\beta^3}{\alpha}} \quad \& \quad 1+2^{3/2}\sqrt[24]{\frac{\alpha^5(1-\alpha)^5}{\alpha(1-\alpha)}}$$

$$\times 1+2\sqrt[8]{\frac{(1-\alpha)^3}{1-\alpha}} \qquad \times 1+2^{1/2}\sqrt[24]{\frac{\alpha(1-\alpha)}{\alpha(1-\alpha)}}$$

$$= 3 \qquad = 5$$

$$F\left\{\frac{1-\sqrt{1-(2-\sqrt{3})^4}}{2}\right\} = e^{-3\pi}$$

$$F\left\{\frac{1-\sqrt{1-(\sqrt{5}-2)^8}}{2}\right\} = e^{-5\pi}$$

$$F\left(\frac{1}{2}-\sqrt{\sqrt{5}-2}\right) = e^{-\pi\sqrt{5}}$$

$$F\left(\frac{8-3\sqrt{7}}{16}\right) = e^{-\pi\sqrt{7}}$$

$$1+\left(\frac{1}{4}\right)^2 p\left(\frac{2-p}{1+2p}\right)^5+\left(\frac{1.5}{4.8}\right)^2 p^2\left(\frac{2-p}{1+2p}\right)^{10}+\&c$$

$$=(1+2p)\left\{1+\left(\frac{1}{4}\right)^2 p^5\frac{2-p}{1+2p}+\left(\frac{1.5}{4.8}\right)^2 p^{10}\left(\frac{2-p}{1+2p}\right)^2+\right.$$

$$\phi(e^{-9\pi}) = \frac{1+\sqrt[3]{2(\sqrt{3}+1)}}{3}\phi(e^{-\pi})$$

$$1+\left(\frac{1}{2}\right)^2\frac{1-11p-p^2}{(1+2p)^2}\sqrt{\frac{1+p^2}{1+2p}}+\&c$$

$$=(1+2p)\left\{1+\left(\frac{1}{2}\right)^2\frac{1-(1+p-p^2)}{2}\sqrt{\frac{1+p^2}{1+2p}}+\right.$$

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$$\frac{\psi^7(x)}{\psi(x^7)} + \frac{\psi^7(-x)}{\psi(-x^7)} + 14 \cdot \frac{f^7(-x^8,-x^{16})}{f(-x^{56},-x^{112})}$$

$$= 8\frac{\psi^7(x^2)}{\psi(x^{14})} + 2\frac{f^7(-x^2,-x^{14})}{f(-x^{14},-x^{28})} + 6\, f(-x^4,-x^{28})$$

$$8\frac{\psi^7(x^2)}{\psi(x^{14})} + \cdots$$

$$\left\{\frac{\phi^2(x)}{\phi(x^7)} - \phi^2(x^7)\right\}^2 \frac{\phi(x^7)}{\phi(x)} - \left\{\phi^2(-x) - \phi^2(x^7)\right\}^2 \frac{\phi(-x^7)}{\phi(-x)}$$

$$= \phi(x^4)\, e^{i\left\{\frac{\pi}{4} - \tan^{-1}\frac{\phi(-x)}{\phi(x)}\right\}}$$

$$\phi^3(x^4)\phi^3(x^{14}) \sin\left\{3\tan^{-1}\frac{\phi(-x^7)}{\phi(x^7)} - \tan^{-1}\frac{\phi(-x)}{\phi(x)}\right\}$$

$$+ 2\phi(x^2)\phi^5(x^{14}) \sin\left\{5\tan^{-1}\frac{\phi(-x)}{\phi(x)} - \tan^{-1}\frac{\phi(-x^7)}{\phi(x^7)}\right\}$$

$$+ \frac{\phi^7(x^{14})}{\phi(x^2)} \sin\left\{7\tan^{-1}\frac{\phi(-x^7)}{\phi(x^7)} + \tan^{-1}\frac{\phi(-x)}{\phi(x)}\right\}$$

$$= \left(\frac{\sqrt{6}-\sqrt{2}-1}{\sqrt{6}+\sqrt{2}+1}\right) = e^{-\pi\sqrt{6}}$$

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$$\frac{1-2\beta}{2\alpha-1} = \frac{f(x)}{f\left(\frac{2}{x}\right)}$$

$$F\frac{(7-4\sqrt{3})(4-\sqrt{15})}{(\sqrt{5}+1)^4} = e^{-\pi\sqrt{15}}$$

$$F\left(\frac{16-7\sqrt{3}\pm\sqrt{15}}{32}\right) = e^{-\pi\sqrt{15}}$$

$$F\left\{(2-\sqrt{3})^4(\sqrt{2}-1)^6\right\} = e^{-3\pi\sqrt{2}}$$

$$F\left(\frac{\sqrt{6}-\sqrt{2}-1}{\sqrt{2}-1}\right)^2 = e^{-\pi\sqrt{6}}$$

7. $\frac{1+\left(\frac{4}{5}\right)^2(1-\alpha)+\alpha}{1+\left(\frac{4}{5}\right)^2\alpha+\alpha} = \frac{1+\left(\frac{4}{5}\right)^2(1-\beta)+\beta}{1+\left(\frac{4}{5}\right)^2\beta+\beta}$

VII. $\sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)} = 1$

III $\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} = 1$

I. $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} = 1$

$$\sqrt{7.\frac{\sqrt[8]{\alpha\beta}-\beta}{\alpha-\sqrt[8]{\alpha\beta}}}$$

$$\frac{\sqrt[8]{\alpha\beta^3}+\sqrt[8]{(1-\alpha)(1-\beta)^3}}{\sqrt[8]{\alpha^3\beta}+\sqrt[8]{(1-\alpha)^3(1-\beta)}}$$

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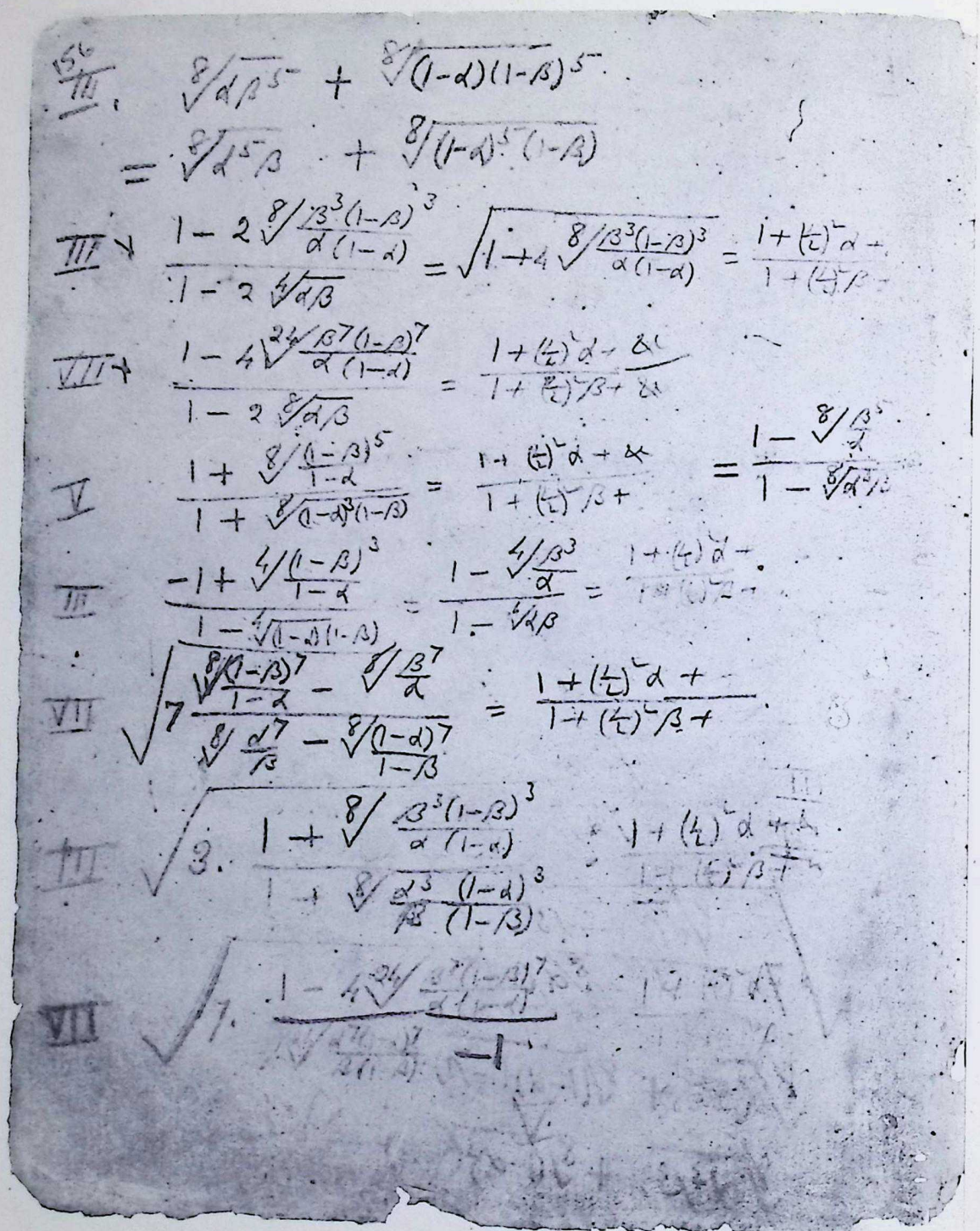

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III $\sqrt{3 \cdot \dfrac{1 - 2\sqrt[8]{\dfrac{\beta^2(1-\beta)^2}{\alpha(1-\alpha)}}}{2\sqrt[8]{\dfrac{\alpha^2(1-\alpha)^2}{\beta(1-\beta)}} - 1}} = \dfrac{1+(\frac{1}{2})^2\alpha + \&c}{1+(\frac{1}{2})^2\beta + \&c}$

Solvable from above

XI $\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} + 2\sqrt[3]{2}\,\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)} = 1$

XXIII $\sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)} + \sqrt[3]{4}\,\sqrt[24]{\alpha\beta(1-\alpha)(1-\beta)} = 1$

V $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 2\sqrt[3]{4}\,\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)} = 1$

III. $\sqrt{\dfrac{\beta}{\alpha}} + \sqrt{\dfrac{1-\beta}{1-\alpha}} - \sqrt{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} = \sqrt{\left(\dfrac{1+(\frac{1}{2})^2\alpha + \&c}{1+(\frac{1}{2})^2\beta + \&c}\right)^2}$

V $\sqrt[4]{\dfrac{\beta}{\alpha}} + \sqrt[4]{\dfrac{1-\beta}{1-\alpha}} - \sqrt[4]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} = \dfrac{1+(\frac{1}{2})^2\alpha + \&c}{1+(\frac{1}{2})^2\beta + \&c}$

IX $\sqrt[8]{\dfrac{\beta}{\alpha}} + \sqrt[8]{\dfrac{1-\beta}{1-\alpha}} - \sqrt[8]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} = \sqrt{\dfrac{1+(\frac{1}{2})^2\alpha + \&c}{1+(\frac{1}{2})^2\beta + \&c}}$

XXV $\sqrt[8]{\dfrac{\beta}{\alpha}} + \sqrt[8]{\dfrac{1-\beta}{1-\alpha}} - \sqrt[8]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 2\sqrt[12]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} = \sqrt{\dfrac{1+(\frac{1}{2})^2\alpha + \&c}{1+(\frac{1}{2})^2\beta + \&c}}$

XIII $\sqrt[4]{\dfrac{\beta}{\alpha}} + \sqrt[4]{\dfrac{1-\beta}{1-\alpha}} - \sqrt[4]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 4\sqrt[6]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} = \dfrac{1+(\frac{1}{2})^2\alpha + \&c}{1+(\frac{1}{2})^2\beta + \&c}$

VII $\sqrt{\dfrac{\beta}{\alpha}} + \sqrt{\dfrac{1-\beta}{1-\alpha}} - \sqrt{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 8\sqrt[3]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} = \left\{\dfrac{1+(\frac{1}{2})^2\alpha + \&c}{1+(\frac{1}{2})^2\beta + \&c}\right.$

XV $\sqrt[16]{\alpha\beta}\left(\sqrt[4]{1+\sqrt{\alpha}}\,\sqrt[4]{1+\sqrt{\beta}} + \sqrt[4]{1-\sqrt{\alpha}}\,\sqrt[4]{1-\sqrt{\beta}}\right)$

$+ \sqrt[16]{(1-\alpha)(1-\beta)}\left(\sqrt[4]{1+\sqrt{1-\alpha}}\,\sqrt[4]{1+\sqrt{1-\beta}} + \sqrt[4]{1-\sqrt{1-\alpha}}\,\sqrt[4]{1-\sqrt{1-\beta}}\right)$

$= \sqrt{2}$

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III $\sqrt[8]{\alpha\beta^5} + \sqrt[8]{(1-\alpha)(1-\beta)^5} + \sqrt[8]{\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}} = 1$

V $\sqrt[8]{\alpha\beta^3} + \sqrt[8]{(1-\alpha)(1-\beta)^3} + \frac{3}{2}\cdot\sqrt[24]{\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}} = 1$

XXXI $\sqrt[32]{\alpha\beta}\left\{\sqrt[8]{(1+\sqrt{\alpha})(1+\sqrt{\beta})}\sqrt{1+\sqrt[3]{\alpha\beta}} + \sqrt{(1-\sqrt{\alpha})(1-\sqrt{\beta})}\right.$

$\left. + \sqrt[8]{(1-\sqrt{\alpha})(1-\sqrt{\beta})}\sqrt{1+\sqrt[4]{\alpha\beta}} + \sqrt{(1+\sqrt{\alpha})(1+\sqrt{\beta})}\right\}$

$\left. + \sqrt[32]{(1-\alpha)(1-\beta)}\right\}\{\alpha c\,\alpha c\cdot\alpha c\} = \sqrt{2}/8$

$F\left(\frac{1}{2} - 3\sqrt{5\sqrt{13}-18}\right) = e^{-\pi\sqrt{13}}$

changing β to $\frac{4\beta}{(1+\beta)^2}$ & q to $1-\beta^2$ we get an equation in $\frac{4\beta(1-\beta)}{1+\beta}$ and the value of β^2 is for $e^{-\pi\sqrt{2n}}$

If α I degree, β 2nd & γ 9th

then $1 + \sqrt[3]{4}\cdot\sqrt[24]{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}} = 3\sqrt{\frac{1+(\frac{1}{2})^2\gamma+\&}{1+(\frac{1}{2})^2\alpha+\&c}}$

III $\sqrt{\alpha(1-\beta)} + \sqrt{\beta(1-\alpha)} = 2\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$

V $\sqrt[4]{\alpha(1-\beta)} + \sqrt[4]{\beta(1-\alpha)} = \sqrt[3]{4}\sqrt[24]{\alpha\beta(1-\alpha)(1-\beta)}$

Let $\alpha = \sin^2(u+v)$ & $\beta = \sin^2(u-v)$

III $\sin 2u = 2\sin v$; V $\sin 2u = \sin v(1+\cos v)$

VII $\sin 2u = 4\sin\frac{v}{2}\sqrt{\cos^2 v + 3\sin^4\frac{v}{2}}$

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$$F\,\frac{1-\sqrt{1-(55\pm 12\sqrt{21})(8-3\sqrt{7})^2}}{2} = e^{-\pi\sqrt{21}}$$ ~~160~~ 159

$$F\,\frac{1-\sqrt{1-\left(\frac{\sqrt{4+\sqrt{7}}-\sqrt[4]{7}}{2}\right)^{24}}}{2} = e^{-7\pi}$$

$$F\,\frac{1-\sqrt{1-(\sqrt{5}-2)^8(2-\sqrt{3})^8\left(\frac{\sqrt{4+\sqrt{15}}\pm\sqrt[4]{15}}{2}\right)^{24}}}{2} = e^{-15\pi}$$

If α is 1st. deg. β 3rd, γ 5th & δ 15th. then

$$\frac{\sqrt[8]{\beta\gamma}-\sqrt[8]{\beta\gamma(1-\beta)(1-\gamma)}}{\sqrt[4]{\alpha\delta}} = \frac{\sqrt[8]{(1-\beta)(1-\gamma)}-\sqrt[8]{\beta\gamma(1-\beta)(1-\gamma)}}{\sqrt[4]{(1-\alpha)(1-\delta)}}$$

$$= \sqrt{\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}\cdot\frac{1+(\frac{1}{2})^2\delta+\&c}{1+(\frac{1}{2})^2\gamma+\&c}}$$

$$\phi(x)\phi(x^9) = \phi^2(x^3)+2x\,\phi(-x^2)\,\psi(x^9)\,(1+x^3)(1+x^9)(1+x^{15})\,\&c$$

$$\frac{1+(\frac{1}{2})^2\beta+(\frac{1.3}{2.4})^2\beta^2+\&c}{\sqrt{1+(\frac{1}{2})^2\alpha+\&c}\,\sqrt{1+(\frac{1}{2})^2\gamma+\&c}} = 1-\sqrt[3]{4}\,\sqrt[24]{\frac{\gamma^3(1-\alpha)^3}{\beta(1-\beta)}} = \sqrt[3]{4}\,\sqrt[24]{\frac{\alpha^3(1-\gamma)^3}{\beta(1-\beta)}}-1$$

where α is 1st, β 3rd & γ 9th deg

$$\therefore \sqrt[8]{\alpha(1-\gamma)}+\sqrt[8]{\gamma(1-\alpha)} = \sqrt[3]{2}\,.\,\sqrt[24]{\beta(1-\beta)}.$$

$$\sqrt{\alpha\gamma}+\sqrt{(1-\alpha)(1-\gamma)}+2\sqrt[3]{4\beta(1-\beta)}$$
$$= 1+8\sqrt[4]{\beta(1-\beta)}\,\sqrt[8]{\alpha\gamma(1-\alpha)(1-\gamma)}$$

$$1+\sqrt[3]{4}\,\sqrt[24]{\frac{\gamma^3(1-\gamma)^3}{\beta(1-\beta)}} = \sqrt{\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\gamma+\&c}}$$

$$1-2\sqrt[3]{4}\,\sqrt[24]{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}}\,\sqrt[24]{\frac{\gamma^3(1-\gamma)^3}{\beta(1-\beta)}} = \frac{1+(\frac{1}{2})^2\beta}{1+(\frac{1}{2})^2\alpha}\cdot\frac{1+\beta}{1+(\frac{1}{2})^2\gamma}$$

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$$\psi(x) - 3x\psi(x^9) = \frac{f(-x)}{(1-x^3)(1-x^9)(1-x^{15})} \&c.$$

$$\phi(x)\phi(x^{15}) - \phi(x^3)\phi(x^5)$$
$$= 2x f(-x^2) f(-x^{30}) \cdot (1+x^3)(1+x^5)(1+x^{15}) \&c$$
$$+ (1+x^5)(1+x^{15})(1+x^{25}) +\&c.$$

$$\phi(x)\phi(x^9) + \phi^2(x^3) = 2\psi(x)\phi(-x^{18})(1+x^3)(1+x^9)(1+$$

$$\phi(x)\phi(x^{15}) + \phi(x^3)\phi(x^5)$$
$$= 2 f(-x^6) f(-x^{10}) \cdot (1+x)(1+x^2)(1+x^5) \&c$$
$$+ (1+x^{11})(1+x^{13})(1+x^{23}) +\&$$

$$\psi(x^3)\psi(x^5) - x\psi(x)\psi(x^{15}) = \frac{f(-x)\, f(-x^{15})}{\chi(-x^3)\,\chi(-x^5)}$$

$$\psi(x^3)\psi(x^5) + x\psi(x)\psi(x^{15}) = \frac{f(-x^3)\, f(-x^5)}{\chi(-x)\,\chi(-x^{15})}$$

$$F\left(\frac{1-\sqrt{1-x^{24}}}{2}\right) = e^{-13\pi}$$

where $x + \frac{1}{x} = \dfrac{\left(\sqrt[8]{3\sqrt{3}+1} + \sqrt[8]{3\sqrt{3}-1}\right)^2 \sqrt[6]{13}}{3\sqrt[3]{2}}$

$$F\left(\frac{1-\sqrt{1-x^{24}}}{2}\right) = e^{-11\pi}$$

where $x + \frac{1}{x} = \dfrac{\sqrt[3]{9\sqrt{3}+1} + \sqrt[3]{9\sqrt{3}-1}}{\sqrt{3}} \cdot \sqrt[6]{\frac{11}{2}}$

$$x^{\frac{3}{8}}\left\{\sqrt{\frac{\sqrt{x}+\frac{1}{\sqrt{x}}+1}{2}} - \sqrt{\frac{\sqrt{x}+\frac{1}{\sqrt{x}}-1}{2}}\right\} = \sqrt{A+1} \pm \sqrt{A}$$

where $A = \frac{3}{2} \cdot \dfrac{1}{\sqrt{\frac{x+\frac{1}{x}+1}{x+\frac{1}{x}-2}} - 1}$

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$$\psi^2(-x) + x\psi^2(-x^5) = \frac{f(x^5)\,\phi(x^5)}{\chi(x)}$$

$$\psi^2(-x) + 5x\psi^2(-x^5) = \frac{\phi^2(x)}{\chi(x)\,\chi(x^5)}$$

III. If $\alpha = \frac{1 \pm \sqrt{1-x^3}}{2}$ & $\beta = \frac{1-\sqrt{1-y^8}}{2}$

then $y\sqrt{2} = \sqrt[8]{x}\left\{\sqrt{1+x+x^2} - x \pm \sqrt{(1-x)\left(2\sqrt{1+x+x^2}+1+x\right)}\right.$

$$\left. \pm \sqrt{(1-x)\left(2\sqrt{1+x+x^2}-1-2x\right)}\right\}$$

$$F\left\{\frac{32 + 7\sqrt{7} - 7\sqrt{3} - (16-\sqrt{21})\sqrt{2(\sqrt{21}-3)}}{64}\right\}$$

$$= e^{-3\pi\sqrt{7}}$$

$$F\,\frac{1-\sqrt{1-(\sqrt{5}-2)^6(4\pm\sqrt{15})^2}}{2} = e^{-2\pi\sqrt{5}}$$

$$F\,\frac{1-\sqrt{1-\frac{4}{\left(\sqrt[3]{1+\sqrt{\frac{11}{27}}}+\sqrt[3]{1-\sqrt{\frac{11}{27}}}\right)12}}}{2} = e^{-\pi\sqrt{11}}$$

$$F\,\frac{1-\sqrt{1-\frac{4}{\left(\sqrt[3]{1+\sqrt{\frac{23}{27}}}+\sqrt[3]{1-\sqrt{\frac{23}{27}}}\right)24}}}{2} = e^{-\pi\sqrt{23}}$$

$$F\,\frac{1-\sqrt{1-\frac{2^6}{\left(\sqrt[3]{1+\sqrt{\frac{19}{27}}}+\sqrt[3]{1-\sqrt{\frac{19}{27}}}\right)24}}}{2} = e^{-\pi\sqrt{19}}$$

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$$F\,\frac{1-\sqrt{1-\left(\frac{\sqrt[3]{1+\sqrt{\frac{31}{27}}}+\sqrt[3]{1-\sqrt{\frac{31}{27}}}}{2^{14}}\right)^{24}}}{2} = e^{-\pi\sqrt{31}}$$

XVII $\sqrt[4]{\frac{\beta}{\alpha}} + \sqrt[4]{\frac{1-\beta}{1-\alpha}} + \sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$

$$-2\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left\{1+\sqrt[8]{\frac{\beta}{\alpha}}+\sqrt[8]{\frac{1-\beta}{1-\alpha}}\right\} = \frac{1+\left(\frac{1}{2}\right)^2\alpha+\ldots}{1+\left(\frac{1}{2}\right)^2\beta+\ldots}$$

$$F\,\frac{1-\sqrt{1-\left(\frac{1}{\sqrt{5-\sqrt{17}}}-\frac{1}{\sqrt{3+\sqrt{17}}}\right)^{24}}}{2} = e^{-\pi\sqrt{17}}$$

XXXIII ? $\sqrt{\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}+\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}-2\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left\{1+\sqrt[8]{\frac{\beta}{\alpha}}+\sqrt[8]{\frac{1-\beta}{1-\alpha}}\right\}}$

$$-2\sqrt[16]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\sqrt[4]{\sqrt[8]{\frac{\beta}{\alpha}}+\sqrt[8]{\frac{1-\beta}{1-\alpha}}+\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}}$$

$$=\sqrt{\frac{1+\left(\frac{1}{2}\right)^2\alpha+\left(\frac{1}{2}\cdot\frac{3}{4}\right)^2\alpha^2+\ldots}{1+\left(\frac{1}{2}\right)^2\beta+\left(\frac{1}{2}\cdot\frac{3}{4}\right)^2\beta^2+\ldots}}$$

IX $\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{1-\beta}{1-\alpha}}+\sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$

$$-4\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left\{1+\sqrt[4]{\frac{\beta}{\alpha}}+\sqrt[4]{\frac{1-\beta}{1-\alpha}}\right\} = \left\{\frac{1+\left(\frac{1}{2}\right)^2\alpha+\ldots}{1+\left(\frac{1}{2}\right)^2\beta+\ldots}\right\}$$

VII $\left\{\sqrt[4]{\frac{(1-\beta)^7}{1-\alpha}}+\sqrt[4]{\frac{\beta^7}{\alpha}}+1\right\}-2\left\{\sqrt[8]{\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}+\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}}+\sqrt[8]{\frac{\beta^7}{\alpha}}\right.$

$$-\sqrt[24]{\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}\left\{\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}}+\sqrt[8]{\frac{\beta^7}{\alpha}}+1\right\}-3\sqrt[12]{\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}=0$$

The ~~rational part of~~ $2x\,\psi\left(x\frac{8m}{m+n}\right)\psi\left(x\frac{8n}{m+n}\right)$

+ its complementary function $= \phi(x)\,\phi(x^{m+n})$

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III $\sqrt[8]{\alpha\beta^5} + \sqrt[8]{(1-\alpha)(1-\beta)^5} = \sqrt{1 - \sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}}$.

V $\sqrt[8]{\alpha\beta^3} + \sqrt[8]{(1-\alpha)(1-\beta)^3} = \sqrt{1 - \sqrt[6]{16\alpha\beta(1-\alpha)(1-\beta)}}$

$= \sqrt{\frac{1+\sqrt{\ }}{2}}$

III $\dfrac{1 + \sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}}}{\sqrt{1 - \sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}}} = \dfrac{1+(\frac{1}{2})^2\alpha+\alpha}{1+(\frac{1}{2})^2\beta+\alpha}$

VII $\dfrac{\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}} - \sqrt[8]{\frac{\beta^7}{\alpha}}}{1 - \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}} = \dfrac{1+(\frac{1}{2})^2\alpha+\alpha}{1+(\frac{1}{2})^2\beta+\alpha}$

$$\frac{\phi^2(e^{-7\pi})}{\phi^2(e^{-\pi})} = \frac{\left(\sqrt{13+\sqrt{7}} + \sqrt{7+3\sqrt{7}}\right)}{14}\sqrt[8]{28}$$

$$\phi(x^3)\phi(x^5) = \phi(-x^2)\phi(-x^{30}) + 2x^2\psi(x)\psi(x^{15}).$$

If I α, 3rd β, 5th γ & 15th δ then

$$\sqrt[8]{\alpha\delta} + \sqrt[8]{(1-\alpha)(1-\delta)} = \sqrt{\frac{1+(\frac{1}{2})^2\beta+\alpha}{1+(\frac{1}{2})^2\alpha+\alpha}\cdot\frac{1+(\frac{1}{2})^2\gamma+\alpha}{1+(\frac{1}{2})^2\delta+\alpha}}$$

$$\sqrt[8]{\beta\gamma} + \sqrt[8]{(1-\beta)(1-\gamma)} = \sqrt{\frac{1+(\frac{1}{2})^2\alpha+\alpha}{1+(\frac{1}{2})^2\beta+\alpha}\cdot\frac{1+(\frac{1}{2})^2\delta+\alpha}{1+(\frac{1}{2})^2\gamma+\alpha}}$$

$$\sqrt[8]{\alpha\delta} - \sqrt[8]{(1-\alpha)(1-\delta)} = \sqrt[8]{\beta\gamma} - \sqrt[8]{(1-\beta)(1-\gamma)}$$

$$\sqrt[8]{\beta\gamma} + \sqrt[8]{(1-\beta)(1-\gamma)} = \sqrt[3]{4}\cdot\sqrt[24]{\frac{\beta^2\gamma^2(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)(1-\delta)}} - 1.$$

$$\sqrt[8]{\alpha\delta} + \sqrt[8]{(1-\alpha)(1-\delta)} = -\sqrt[3]{4}\cdot\sqrt[24]{\frac{\alpha^2\delta^2(1-\alpha)^2(1-\delta)^2}{\beta\gamma(1-\beta)(1-\gamma)}} + 1.$$

VII $\dfrac{\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}} - 1}{\sqrt[8]{\frac{(1-\beta)^7}{1-\alpha}} - \sqrt[8]{\frac{\beta^7}{\alpha}}} = \sqrt[8]{\alpha\beta}$.

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$$\sqrt[3]{\alpha\delta} + \sqrt[3]{(1-\alpha)(1-\delta)} + 2\sqrt[12]{\alpha\delta(1-\alpha)(1-\delta)}$$
$$+ \sqrt[3]{\beta\gamma} + \sqrt[3]{(1-\beta)(1-\gamma)} + 2\sqrt[12]{\beta\gamma(1-\beta)(1-\gamma)}$$
$$= 1 + \left\{1 + 2\sqrt[6]{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}\right\}^2.$$

$$P = \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{1-\beta}{1-\alpha}} + \sqrt[3]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$$

$$Q = \sqrt[3]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} \cdot \left\{\sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{1-\beta}{1-\alpha}} + 1\right\}$$

$$R = \sqrt[3]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$$

17. $\sqrt{P^2 - 4Q} = K.$

19. $P + R^{\frac{1}{3}} - \sqrt{4Q + 4PR^{\frac{1}{3}} + 13R^{\frac{2}{3}}} = K.$

V.
$$\frac{m + 2(m-n)\sqrt[3]{2}\sqrt[24]{\frac{\beta^2(1-\beta)^2}{\alpha(1-\alpha)}} + n\sqrt[3]{4}\sqrt[12]{\frac{\beta^2(1-\beta)^2}{\alpha(1-\alpha)}}}{m - n\sqrt[8]{16\alpha\beta(1-\alpha)(1-\beta)}}$$
$$= \frac{1 - \sqrt[3]{2}\sqrt[24]{\frac{\beta^2(1-\beta)^2}{\alpha(1-\alpha)}} - \sqrt[3]{4}\sqrt[12]{\frac{\beta^2(1-\beta)^2}{\alpha(1-\alpha)}}}{\sqrt{1 - 8\sqrt[4]{16\alpha\beta(1-\alpha)(1-\beta)} + \sqrt[3]{160\beta(1-\beta)(1-\beta)}}} = \frac{1 + (\frac{1}{2})^2\alpha + \beta}{1 + (\frac{1}{2})^2\beta + \alpha}$$

VI.
$$\frac{1 + 8\sqrt[3]{2}\sqrt[24]{\frac{\beta^{11}(1-\beta)^{11}}{\alpha(1-\alpha)}}}{\left\{\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)}\right\}\left\{\sqrt[3]{\alpha\beta} - \sqrt[3]{(1-\alpha)(1-\beta)} + \sqrt{2 + 2\sqrt[3]{\alpha\beta}} + 2\sqrt{1 - \cdots}\right.}$$
$$= \frac{1 + (\frac{1}{2})^2\alpha + (\frac{1}{2})^2\alpha^2 + \cdots}{1 + (\frac{1}{2})^2\beta + (\frac{1}{2})^2\beta^2 + \cdots}$$

VII. $$\sqrt[3]{(1-\alpha)^7} + \sqrt[3]{\beta^7} + 2\sqrt[24]{\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}$$
$$= \cdots \left\{1 + \cdots\right\}^2.$$

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1st α, 5th β, 7th γ & 35th δ

$$\left\{\sqrt[4]{\alpha\delta} + \sqrt[4]{(1-\alpha)(1-\delta)} + 2\sqrt[3]{2}\,\sqrt[12]{\alpha\delta(1-\alpha)(1-\delta)}\right\}$$

$$\times\left\{\sqrt[8]{\beta\gamma} + \sqrt[4]{(1-\beta)(1-\gamma)} + 2\sqrt[3]{2}\,\sqrt[12]{\beta\gamma(1-\beta)(1-\gamma)}\right\}$$

$$= 1 - 4\sqrt[3]{2}\,\sqrt[24]{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}\left\{\sqrt[8]{\alpha\beta\gamma\delta} + \sqrt[8]{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}\right\}$$

$$\phi(x)\,\phi(x^{27}) - \phi(-x)\,\phi(-x^{27}) = 4x\,f(-x^6)\,f(-x^{18}) + 4x^7\,\psi(x^2)\,\psi(x^{54})$$

$$\phi(x)\,\phi(x^{35}) - \phi(-x)\,\phi(-x^{35}) = 4x\,f(-x^{10})\,f(-x^{14}) + 4x^9\,\psi(x^2)\,\psi(x^{70})$$

$$\phi(x^5)\,\phi(x^7) - \phi(-x^5)\,\phi(-x^7) = 4x^3\,\psi(x^{10})\,\psi(x^{14}) - 4x^3\,f(-x^2)\,f(-x^{70})$$

$$\phi(x)\,\phi(x^{63}) - \phi(x^7)\,\phi(x^9) = 2x\,f(x^3)\,f(x^{21})$$

If $\alpha, \beta, \gamma, \delta = 1, 3, 13, 39$ or $1, 5, 11, 55$ or $1, 7, 9, 63$

then
$$\frac{1 + \sqrt[4]{(1-\alpha)(1-\delta)} + \sqrt[4]{\alpha\delta}}{1 + \sqrt[4]{(1-\beta)(1-\gamma)} + \sqrt[4]{\beta\gamma}} = \frac{\sqrt[8]{(1-\alpha)(1-\delta)} - \sqrt[8]{\alpha\delta}}{\sqrt[8]{(1-\beta)(1-\gamma)} - \sqrt[8]{\beta\gamma}}$$

$$= \sqrt{\frac{1+(\frac{1}{2})^2\beta + \&c}{1+(\frac{1}{2})^2\alpha + \&c} \cdot \frac{1+(\frac{1}{2})^2\gamma + \&c}{1+(\frac{1}{2})^2\delta + \&c}}$$

$$= \frac{\sqrt[8]{\alpha\delta} \pm \sqrt[8]{\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[8]{\beta\gamma} - \sqrt[8]{\beta\gamma(1-\beta)(1-\gamma)}} \quad \left(\begin{array}{l} + \text{ in the I 2 cases } \& \\ - \text{ in } 1, 7, 9, 63 \end{array}\right)$$

$$\psi(x^7)\,\psi(x^9) - x^6\,\psi(x)\,\psi(x^{63}) = f(-x^6)\,f(-x^{42})\ldots$$

$$\phi(x)\,\phi(x^{135}) = \phi(-x^{10})\,\phi(-x^{54}) + 2x^4\,\psi(x^5)\,\psi(x^{27}) + 2x\,f(x^9)\,f(x^{[illegible]})$$

$$\phi(x^5)\,\phi(x^{27}) = \phi(-x^2)\,\phi(-x^{270}) + 2x^{17}\,\psi(x)\,\psi(x^{[illegible]}) + 2x^2\,f(x^3)\,f(x^{[illegible]})$$

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III $M^2\sqrt{\alpha(1-\alpha)} + \sqrt{\beta(1-\beta)} = 2\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$

V $M\sqrt[4]{\alpha(1-\alpha)} + \sqrt[4]{\beta(1-\beta)} = \sqrt[4]{4}\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}$

III $\begin{cases} M\sqrt{1-\alpha} + \sqrt{1-\beta} = 2\sqrt[8]{(1-\alpha)(1-\beta)} \\ M\sqrt{\alpha} - \sqrt{\beta} = 2\sqrt[8]{\alpha\beta} \end{cases}$

II $M\sqrt{1-\alpha} + \sqrt{\beta} = 1$ | II $M^2\sqrt{1-\alpha} + \beta = 1$

IV $\sqrt{M}\sqrt[4]{1-\alpha} + \sqrt[4]{\beta} = 1$ | VIII $\sqrt{M}\sqrt[8]{1-\alpha} + \sqrt[4]{\beta} = 1$

II $M^2 = 2\cdot\dfrac{1+\beta}{1+(1-\alpha)} = 2\cdot\dfrac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}}$

IV $M = 2\cdot\dfrac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}} = 2\cdot\dfrac{1+\sqrt[4]{\beta}}{1+\sqrt[4]{1-\alpha}}$.

XVI $\sqrt{M} = 2\cdot\dfrac{1+\sqrt[4]{\beta}}{1+\sqrt[4]{1-\alpha}}$.

IV $\sqrt[3]{\dfrac{\beta}{\alpha}} + \sqrt[3]{\dfrac{1-\beta}{1-\alpha}} - \dfrac{4}{M}\sqrt[3]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} = M$.

III $\sqrt[3]{\dfrac{\beta}{\alpha}} + \sqrt[3]{\dfrac{1-\beta}{1-\alpha}} - \dfrac{7}{M}\sqrt[3]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 3\sqrt[6]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} = M$.

IX $M = 3\cdot\dfrac{1+2\sqrt[3]{\beta}}{1+2\sqrt[3]{1-\alpha}}$.

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Equations for $1 + \frac{1.3}{4^2}x + \frac{1.3.5.7}{4^2.8^2}x^2 + \&c$ 163

(3) $\sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{1-\beta}{1-\alpha}} - \frac{9}{M^2}\sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} = M^2$

(5) $\sqrt[4]{\frac{\beta}{\alpha}} + \sqrt[4]{\frac{1-\beta}{1-\alpha}} - \frac{5}{M}\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} = M$

(9) $\sqrt[8]{\frac{\beta}{\alpha}} + \sqrt[8]{\frac{1-\beta}{1-\alpha}} - \frac{3}{\sqrt{M}}\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} = \sqrt{M}.$

(7) $\sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{1-\beta}{1-\alpha}} - \frac{49}{M^2}\sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 8\sqrt[6]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left(\sqrt[6]{\frac{\beta}{\alpha}} + \sqrt[6]{\frac{1-\beta}{1-\alpha}}\right)$

$= M^2$

(13) $\sqrt[4]{\frac{\beta}{\alpha}} + \sqrt[4]{\frac{1-\beta}{1-\alpha}} - \frac{13}{M}\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 4\sqrt[12]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left(\sqrt[12]{\frac{\beta}{\alpha}} + \sqrt[12]{\frac{1-\beta}{1-\alpha}}\right) = M$

(25) $\sqrt[8]{\frac{\beta}{\alpha}} + \sqrt[8]{\frac{1-\beta}{1-\alpha}} - \frac{5}{\sqrt{M}}\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 2\sqrt[24]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left(\sqrt[24]{\frac{\beta}{\alpha}} + \sqrt[24]{\frac{1-\beta}{1-\alpha}}\right) = \sqrt{M}$

(2). $1 + \frac{1.3}{4^2}\left\{1 - \left(\frac{1-t}{1+3t}\right)^2\right\} + \&c$

$= \sqrt{1+3t}\left\{1 + \frac{1.3}{4^2}t^2 + \frac{1.3.5.7}{4^2.8^2}t^4 + \right\}$

(3). If $\alpha = \frac{64t}{(3+6t-t^2)^2}$ $\beta = \frac{64t^3}{(27-18t-t^2)^2}$

then $\sqrt{1 - \frac{2}{3}t - \frac{t^2}{27}}\left\{1 + \frac{1.3}{4^2}\alpha + \right\}$

$= \sqrt{1 + 2t - \frac{t^2}{3}}\left\{1 + \frac{1.3}{4^2}\beta + \right\}.$

$1 + 12\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \&c\right) = \phi^2(x)\phi^2(x^3) \cdot \frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)}}{2}$

$1 + 4\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \&c\right) = \phi^2(x)\phi^2(x^7) \cdot \frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)}}{2}$

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$$P = 1 - \sqrt[8]{\alpha\beta} - \sqrt[8]{(1-\alpha)(1-\beta)}$$

$$Q = 4\left\{\sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)} - \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\right\}$$

$$R = 4\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}.$$

7. $P = 0$. 23. $P - R^{\frac{1}{3}} = 0$.

39. ~~$P^5 - R(6P^2+Q) + \frac{R^2}{P} + \frac{R-P^2}{P^3-R} = 0$.~~

55. ~~$P^7 - R(18P^4+9P^2Q+Q^2)$~~ $+ 10R^2P + \frac{QR^2}{P}$

$$+ \frac{R^2P^2Q}{P^3-R} = 0.$$

71. $P^3 - R^{\frac{1}{3}}(4P^2+Q) + 2PR^{\frac{2}{3}} - R = 0$.

~~87. $P^{11} -$~~

15. $P^2 - Q + \frac{R}{P} = 0$.

31. $P^2 - Q - \sqrt{PR} = 0$

47. $P^2 - Q - PR^{\frac{1}{3}} - 2R^{\frac{2}{3}} = 0$.

~~63~~. 95. $(P^2-Q)^2 - PR^{\frac{1}{3}}(5-P^2-4Q) + QR^{\frac{2}{3}}$

$$- R\left(P + \frac{Q}{P}\right) - 5R^{\frac{4}{3}} = 0$$

3. $P = 0$ 11. $P - R^{\frac{1}{3}} = 0$. 19. $P^5 - 7P^2R - QR = 0$

35. $P^3 - R^{\frac{1}{3}}(5-P^2+Q) + 2R^{\frac{2}{3}}P - R - \frac{R^{\frac{4}{3}}}{P} = 0$.

~~59. P^5~~ 27. $P^7 - R(29P^4 + 11P^2Q + Q^2) - 17R^2P$

$$- 3\frac{R^2}{P^2}\cdot(PQ+R) = 0$$

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7, 23 &c
$$\begin{cases} P = x f(x^7, x^{14}) - x^{\frac{n+1}{8}} \psi(x^2) \\ Q = \{\phi(-x^2) + 4x^{\frac{n+1}{8}} f(x^7, x^{14})\} \psi(x^2) \\ R = x^{\frac{n+1}{8}} f^3(-x^4) \end{cases}$$

3, 11 &c
$$\begin{cases} P = x\psi(x^8) - x^{\frac{n+1}{4}} \psi(x^2) \\ Q = \phi^2(-x^2) + 16 x^{\frac{n+5}{4}} \psi(x^2) \psi(x^8) \\ R = x^{\frac{n+1}{4}} f^3(-x^2) \end{cases}$$

1, 5 &c
$$\begin{cases} P = \{x\psi(x^4) - 2x^{\frac{n+1}{2}} \phi(x^2)\} \psi(x^4) \\ Q = \phi^4(-x^2) + 128 x^{\frac{n+3}{2}} \psi^2(x^2) \psi^2(x^4) \\ R = x^{\frac{n+1}{2}} f^6(-x^4) \end{cases}$$

15, 31 &c.
$$\begin{cases} P = f(x^6, x^{10}) + x^{\frac{n+1}{8}} \psi(x^2) \\ Q = \{\phi(-x^2) + 4x^{\frac{n+1}{8}} f(x^6, x^{10})\} \psi(x^2) \\ R = x^{\frac{n+1}{8}} f^3(-x^4) \\ P_2 = P^2 - Q = x^2 f^2(x^7, x^{14}) - 2x^{\frac{n+1}{8}} f(x^6, x^{10}) \\ \qquad \times \psi(x^2) + x^{\frac{n+1}{4}} \psi^2(x^2) \end{cases}$$

$P = 1 + \sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)}$
$Q = 4\{\sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)} + \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\}$
$R = 4\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$
$P = 1 - \sqrt[8]{\alpha\beta} - \sqrt[8]{(1-\alpha)(1-\beta)}$
$Q = 4.$
$R = 4.$

$1 - \sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}$
16.
16.
$1 - \sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}$
64.
32.

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$P - R^{\frac{1}{3}} = 0.$ 9. $P^{\frac{5}{3}} - R^{\frac{1}{3}}(14P^2+Q) - 3\frac{R^2}{P} = 0.$

13. $\sqrt{P}(P^3+8R) - \sqrt{R}(14P^2+Q) = 0.$

17. $P^3 - R^{\frac{1}{3}}(10P^2+Q) + 18R^{\frac{1}{3}}P + 12R = 0.$

29. $\sqrt{P}(P^2 + 17R^{\frac{1}{3}}P - 9R^{\frac{2}{3}})$
$- \sqrt[3]{R}(9P^2 + Q - 13R^{\frac{1}{3}}P + 15R^{\frac{2}{3}}) = 0.$

$P^{\frac{5}{3}} - R^{\frac{1}{3}}(14P^4 + 17P^2Q + Q^2) + R^{\frac{2}{3}}P(19P^2+Q)$
$+ 6R(7P^2+Q) + 4R^{\frac{4}{3}}P - 3R^{\frac{5}{3}} = 0.$

21.

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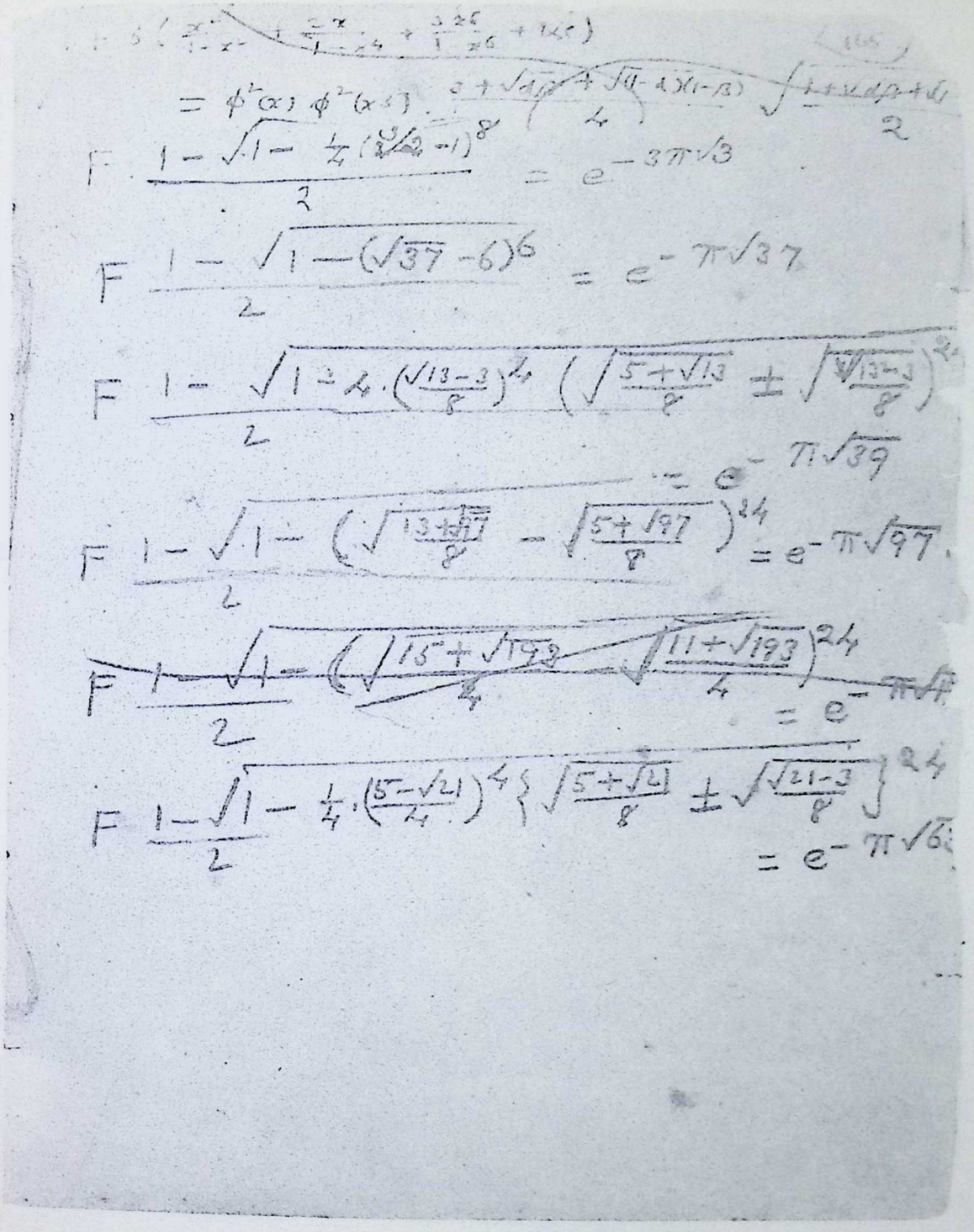

$$\cancel{F\,\delta\left(\frac{x^2}{1-x^2}+\frac{2x}{1-x^4}+\frac{3x^6}{1-x^6}+\cdots\right)}$$

$$= \phi^2(x)\,\phi^2(x^3)\,\frac{\cancel{\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}}{\cancel{4}}\,\cancel{\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{\ldots}}{2}}}$$

$$F\,\frac{1-\sqrt{1-\frac{1}{4}\left(\frac{3}{2}-1\right)^8}}{2} = e^{-3\pi\sqrt{3}}$$

$$F\,\frac{1-\sqrt{1-(\sqrt{37}-6)^6}}{2} = e^{-\pi\sqrt{37}}$$

$$F\,\frac{1-\sqrt{1-4\cdot\left(\frac{\sqrt{13}-3}{8}\right)^4\left(\sqrt{\frac{5+\sqrt{13}}{8}}\pm\sqrt{\frac{\sqrt{13}-3}{8}}\right)^{24}}}{2} = e^{-\pi\sqrt{39}}$$

$$F\,\frac{1-\sqrt{1-\left(\sqrt{\frac{13+\sqrt{97}}{8}}-\sqrt{\frac{5+\sqrt{97}}{8}}\right)^{24}}}{2} = e^{-\pi\sqrt{97}}$$

$$\cancel{F\,\frac{1-\sqrt{1-\left(\sqrt{\frac{15+\sqrt{193}}{4}}-\sqrt{\frac{11+\sqrt{193}}{4}}\right)^{24}}}{2} = e^{-\pi\sqrt{\ldots}}}$$

$$F\,\frac{1-\sqrt{1-\frac{1}{4}\cdot\left(\frac{5-\sqrt{21}}{4}\right)^4\left\{\sqrt{\frac{5+\sqrt{21}}{8}}\pm\sqrt{\frac{\sqrt{21}-3}{8}}\right\}^{24}}}{2} = e^{-\pi\sqrt{6\ldots}}$$

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$\phi(-x)\phi(-x^{95}) + 4x f(x^5) f(x^{19})$

$= \phi(x)\phi(x^{95}) + 4x^6 f(-x^3, -x^{85}) f(-x^{285}, -x^{475})$

$\phi(-x)\phi(-x^{63}) + 4x f(x^3) f(x^{21})$

$= \phi(x)\phi(x^{63}) + 4x^4 f(-x^3, -x^{15}) f(-x^{189}, -x^{315})$

$\phi(-x)\phi(-x^{143}) + 4x f(x^{11}) f(x^{13})$

$= \phi(x)\phi(x^{143}) - 4x^9 f(-x^3, -x^5) f(-x^{429}, -x^{715})$

7. $\sqrt[4]{\frac{(1-\beta)^3}{1-\alpha}} + \sqrt[4]{\frac{\beta^3}{\alpha}} - \sqrt[4]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}} = \frac{M_2}{2}\left\{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}\right\}$

11. $\sqrt[8]{\frac{(1-\beta)^3}{1-\alpha}} - \sqrt[8]{\frac{\beta^3}{\alpha}} - \sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}} = \frac{M}{2}\left\{1 + \sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right\}$

$= M\sqrt{\frac{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}}{2}}$

$M - \frac{3}{M} = 2\left(\sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}\right)$

$M - \frac{5}{M} = 4\left(\sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}\right) \Big/ \sqrt{\frac{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}}{2}}$

$M - \frac{7}{M} = 2\left(\sqrt[8]{\alpha\beta} - \sqrt[8]{(1-\alpha)(1-\beta)}\right)\left(2 + \sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right)$

$M - \frac{11}{M} = 2\left(\sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}\right)\left(4 + \sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right)$

$\cos 2u = (2\cos u - 1)\sqrt{4\cos u - 3}$

$\frac{\sqrt{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}}}{2} = 1 - \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$

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1, 3, 5, 15

$$\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} = -\sqrt{\frac{1+(\tfrac{1}{2})^2\alpha}{1+(\tfrac{1}{2})^2\beta}\cdot\frac{1+(\tfrac{1}{2})^2\gamma}{1+(\tfrac{1}{2})^2\delta}}$$

$$\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} - \sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} = \sqrt{\frac{1+(\tfrac{1}{2})^2\beta}{1+(\tfrac{1}{2})^2\alpha}\cdot\frac{1+(\tfrac{1}{2})^2\gamma}{1+(\tfrac{1}{2})^2\delta}}$$

$$\sqrt[8]{\alpha\beta\gamma\delta} + \sqrt[8]{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)} + \sqrt[3]{2}\cdot\sqrt[24]{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)} = 1$$

1, 5, 25

$$\frac{1+\sqrt[3]{4}\cdot\sqrt[24]{\dfrac{\beta^{10}(1-\beta)^{10}}{\alpha\gamma(1-\alpha)(1-\gamma)}}}{1+\sqrt[3]{4}\cdot\sqrt[24]{\dfrac{\alpha^5\gamma^5(1-\alpha)^5(1-\gamma)^5}{\beta^2(1-\beta)^2}}} = \frac{\phi^2(x)\,\phi^2(x^{-1})}{\phi^4(x^{-1})},$$

1, 3, 13, 39 or 1, 5, 7, 35.

$$\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} + 2\sqrt[12]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$

$$= \cdot \pm \sqrt{\frac{1+(\tfrac{1}{2})^2\alpha+\alpha^{-1}}{1+(\tfrac{1}{2})^2\beta+\alpha^{-1}}\cdot\frac{1+(\tfrac{1}{2})^2\delta+\alpha^{-1}}{1+(\tfrac{1}{2})^2\gamma+\alpha^{-1}}}$$ + in 13 α - in %.

$$\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} - \sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} + 2\sqrt[12]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}$$

$$= \sqrt{\frac{1+(\tfrac{1}{2})^2\delta+\alpha}{1+(\tfrac{1}{2})^2\alpha+\alpha}\cdot\frac{1+(\tfrac{1}{2})^2\beta+\alpha}{1+(\tfrac{1}{2})^2\delta+\alpha}}$$

1, 3, 9

$$\sqrt[4]{\frac{\beta^2}{\alpha\gamma}} + \sqrt[4]{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}} - \sqrt[4]{\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}} = -3\cdot\frac{1+(\tfrac{1}{2})^2\alpha}{1+(\tfrac{1}{2})^2\beta}\cdot\frac{1+(\tfrac{1}{2})^2\gamma}{1+(\tfrac{1}{2})^2\beta}$$

$$\sqrt[4]{\frac{\alpha\gamma}{\beta^2}} + \sqrt[4]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}} - \sqrt[4]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}} = \frac{1+(\tfrac{1}{2})^2\beta+}{1+(\tfrac{1}{2})^2\alpha+}\cdot$$

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1, 3, 7, 21.

$$\sqrt[4]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[4]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[4]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} + \sqrt[4]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$

$$= \sqrt{\frac{1+(\frac{t}{\alpha})^2\alpha+\delta x^2}{1+(\frac{t}{\alpha})^2\beta+\delta x}} \cdot \frac{1+(\frac{t}{\alpha})^2\beta+\alpha}{1+(\frac{t}{\alpha})^2\gamma+\alpha}$$

$$\sqrt[4]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[4]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} - \sqrt[4]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} + \sqrt[4]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}$$

$$= \sqrt{\frac{1+(\frac{t}{\alpha})^2\beta+\alpha}{1+(\frac{t}{\alpha})^2\alpha+\alpha}} \cdot \frac{1+(\frac{t}{\alpha})^2\gamma+\alpha}{1+(\frac{t}{\alpha})^2\delta+\alpha}$$

1, 3, 9, 27.

$$\sqrt[4]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[4]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} + \sqrt[4]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} - 2\sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}\left\{1+\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}}+\sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}}\right\} = -3 \cdot \frac{1+(\frac{t}{\alpha})^2\alpha+\alpha}{1+(\frac{t}{\alpha})^2\beta+\alpha} \times \frac{1+(\frac{t}{\alpha})^2\delta+\alpha}{1+(\frac{t}{\alpha})^2\gamma+\alpha}$$

$$\sqrt[4]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[4]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} + \sqrt[4]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} - 2\sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}\left\{1+\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}}+\sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}}\right\} = \frac{1+(\frac{t}{\alpha})^2\beta+\alpha}{1+(\frac{t}{\alpha})^2\alpha+\alpha} \times \frac{1+(\frac{t}{\alpha})^2\gamma+\alpha}{1+(\frac{t}{\alpha})^2\delta+\alpha}$$

1, 5, 25

$$\sqrt[4]{\frac{\beta^2}{\alpha\gamma}} + \sqrt[4]{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}} + \sqrt[4]{\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}} - 2\sqrt[8]{\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}} \times \left\{1+\sqrt[8]{\frac{\beta^2}{\alpha\gamma}}+\sqrt[8]{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}}\right\} = 5 \cdot \frac{1+(\frac{t}{\alpha})^2\alpha+\alpha}{1+(\frac{t}{\alpha})^2\beta+\alpha} \cdot \frac{1+(\frac{t}{\alpha})^2\gamma+\alpha}{1+(\frac{t}{\alpha})^2\beta+\alpha}$$

$$\sqrt[4]{\frac{\alpha\gamma}{\beta^2}} + \sqrt[4]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}} + \sqrt[4]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}} - 2\sqrt[8]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}} \times \left\{1+\sqrt[8]{\frac{\alpha\gamma}{\beta^2}}+\sqrt[8]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}}\right\} =$$

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1, 5, 25

$$\sqrt[8]{\frac{\alpha\gamma}{\beta^2}} + \sqrt[8]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}} + \sqrt[8]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}}$$

$$= \frac{\left\{1 + \left(\frac{\gamma}{2}\right)^2\beta + \right\}}{\sqrt{\left\{1 + \left(\frac{\alpha}{2}\right)^2\alpha + \right\}}\left\{1 + \left(\frac{\alpha}{2}\right)^2\beta + \right\}}$$

1, 3, 9, 27

$$\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} + \sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}$$

$$= \sqrt{\frac{1 + \left(\frac{\gamma}{2}\right)^2\beta +}{1 + \left(\frac{\alpha}{2}\right)^2\alpha +}} \cdot \frac{1 + \left(\frac{\alpha}{2}\right)^2\gamma +}{1 + \left(\frac{\alpha}{2}\right)^2\delta +}$$

N.B. For 1, 5, 7, 35 Same & for 1, 3, 11, 33 $4R^{\frac{3}{3}}$ instead of $2R^{\frac{3}{3}}$.

For 1, 3, 9, 27 :- $P^5 - R(11P^2+Q) + 9\frac{R^2}{P} + 6\frac{R^3}{P^4} = 0.$

1, 7, 21, 3 or 1, 3, 33, 11.

$$\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} - 2\sqrt[16]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$

$$= \sqrt{\frac{1+\left(\frac{\beta}{2}\right)^2\alpha}{1+\left(\frac{\beta}{2}\right)^2\beta}} \cdot \frac{1+\left(\frac{\gamma}{2}\right)^2\delta}{1+\left(\frac{\gamma}{2}\right)^2\gamma}$$

1, 3, 15, 5

$$\sqrt[4]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[4]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[4]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} - 4\sqrt[6]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$

$$= \frac{1+\left(\frac{\beta}{2}\right)^2\alpha}{1+\left(\frac{\beta}{2}\right)^2\beta} \cdot \frac{1+\left(\frac{\gamma}{2}\right)^2\delta}{1+\left(\frac{\gamma}{2}\right)^2\gamma}$$

For 1, 3, 7, 21 the same eq^n : 1, 3, 3, 9 ; $P^2 = Q + \frac{3R}{P}$ =

1, 5, 5, 25 ; $P^3 - R^{\frac{1}{3}}(5P^2+Q) - R^{\frac{2}{3}}P + 3R - \frac{R^{\frac{4}{3}}}{P} = 0$

1, 3, 17, 51 ; $P^3 - R^{\frac{1}{3}}(7P^2+Q) + 13R^{\frac{2}{3}}P - 19R = 0.$

1, 5, 11, 55 ; $P^3 - R^{\frac{1}{3}}(4P^2+Q) - R^{\frac{2}{3}}P + 4R = 0.$

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37. $P^3 - R^{\frac{1}{3}}(7P^2+Q) - 3PR^{\frac{2}{3}} - 25R - M(19P^2 - \&c) + \&c$

$$\frac{1+2\{\sqrt[3]{\alpha\delta} + \sqrt[3]{(1-\alpha)(1-\delta)}\}}{1+2\{\sqrt[3]{\beta\gamma} + \sqrt[3]{(1-\beta)(1-\gamma)}\}} = \frac{1+\frac{1\cdot2}{5^2}\beta+\&c}{1+\frac{1\cdot2}{5^2}\alpha+\&c} \cdot \frac{1+\frac{1\cdot2}{5^2}\gamma+\&c}{1+\frac{1\cdot2}{5^2}\delta+\&c}$$

$$1 - \sqrt[3]{\alpha\delta} - \sqrt[3]{(1-\alpha)(1-\delta)} = 3\sqrt[6]{\beta\gamma(1-\beta)(1-\gamma)} \cdot \frac{1+\frac{1\cdot2}{5^2}\beta+\&c}{1+\frac{1\cdot2}{5^2}\alpha+\&c} \cdot \frac{1+\frac{1\cdot2}{5^2}\gamma+\&c}{1+\frac{1\cdot2}{5^2}\delta+\&c}$$

$\sqrt{2}$. $\sqrt{e}(\sqrt{2}-1)^2 = e^{-\pi\sqrt{2}}$

$\sqrt{6}$. $(2-\sqrt{3})^2(\sqrt{3}\pm\sqrt{2})^2$

$\sqrt{10}$ $(\sqrt{10}-3)^2(3\pm\sqrt{8})^2$

$\sqrt{18}$ $(5\sqrt{2}-7)^2(7\pm4\sqrt{3})^2$

$\sqrt{22}$ ~~$(\sqrt{66}-\sqrt{65})^2(\sqrt{65}-\sqrt{64})^2$~~ $(10-3\sqrt{11})^2(3\sqrt{11}\pm7\sqrt{2})^2$

$\sqrt{70}$ $(6-\sqrt{35})^2(15-4\sqrt{14})^2(8-3\sqrt{7})^2(3\sqrt{14}-5\sqrt{5})^2$

$\sqrt{58}$. $(13\sqrt{58}-99)^2(99\pm70\sqrt{2})^2$.

$\sqrt{30}$ $(2-\sqrt{3})^2(5-2\sqrt{6})^2(4-\sqrt{15})^2(\sqrt{6}-\sqrt{5})^2$

Let $\alpha\beta = 16k(1-k)^3$ & $(1-\alpha)(1-\beta) = 16k^3(1-k)$

$\therefore \frac{p(1-p)}{1+2p} = 2k$. then $F\left(p^3, \frac{2+p}{1+2p}\right) = e^{-\pi\sqrt{3x}}$

$4k(1-k) = h$

$$\frac{1-\sqrt{1-4h^2}}{2}(1-2K \pm \sqrt{1-K}\cdot\sqrt{1-4K})^2 = e^{-\frac{1}{3}\pi\sqrt{3}}$$

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$$= \frac{1 - \sqrt{1 - (10 \pm 3\sqrt{11})^2 (2-\sqrt{3})^6}}{2} = e^{-\pi\sqrt{33}}$$

$$= \frac{1 - \sqrt{1 - \left(\frac{\sqrt{5}+1}{2}\sqrt[3]{10} + \frac{\sqrt{5}-1}{2}\sqrt[3]{4\sqrt{5}-\sqrt{5}-1}\right)^{24}}}{2}$$

$$= e^{-5\pi\sqrt{3}}.$$

$$= \frac{1 + \sqrt{1 - \left(\frac{\sqrt{5}+1}{2}\sqrt[3]{4\sqrt{5}} - \frac{\sqrt{5}-1}{2}\sqrt[3]{10} + \sqrt{5}-1\right)^{24}}}{2}$$

$$= e^{-\frac{\pi}{5}\sqrt{3}}.$$

$$\sqrt[8]{1 \pm \sqrt{1-(\sqrt{5}-2)^8}} = \frac{\sqrt{5}-1}{2} \cdot \frac{\sqrt[4]{5} \pm 1}{\sqrt{2}}$$

$$\sqrt[8]{1 \pm \sqrt{1-(2-\sqrt{3})^4}} = \frac{\sqrt{2}\cdot\sqrt[4]{3} \pm (\sqrt{3}-1)}{2\sqrt[4]{2}}.$$

If $a = \sqrt[4]{60}$, $b = 2 - \sqrt{3} + \sqrt{5}$ & $RC = 1 + \frac{a+b}{a-b}\sqrt{5}$

then $(\sqrt{c^2+1} - c)\sqrt[5]{e^{6\pi}} = \frac{1}{1+} \frac{e^{-6\pi}}{1+} \frac{e^{-12\pi}}{1+} \&c$

If $RC = 1 + \frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}\sqrt{5}$ then $\frac{e^{-\frac{4\pi}{5}}}{1+} \frac{e^{-4\pi}}{1+} \frac{e^{-8\pi}}{1+} \&c$

If $a = 3 + \sqrt{2} + \sqrt{5}$ & $b = \sqrt[4]{20}$. then $\frac{e^{-\frac{8\pi}{5}}}{1+} \frac{e^{-8\pi}}{1+} \frac{e^{-16\pi}}{1+} \&c$

If $a = \sqrt[4]{5}(4 - \sqrt{2})$ & $b = 1 + \sqrt{2} + \sqrt{5}$
$- \sqrt[4]{2}(3 - \sqrt{2} + \sqrt{5} - \sqrt{10})$

then $\frac{e^{-\frac{16\pi}{5}}}{1+} \frac{e^{-16\pi}}{1+} \frac{e^{-32\pi}}{1+} \&c$

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$$\phi(e^{-45\pi}) = \frac{3+\sqrt{5}+(\sqrt{3}+\sqrt{5}-\sqrt{60})\sqrt[3]{2+\sqrt{3}}}{3\sqrt{10+2\sqrt{5}}}\,\phi(e^{-\pi}).$$

$\frac{\beta(1-\beta)}{1+\beta} = u$ then $\frac{1}{4}\left(u+\frac{1}{u}-2\right)$

1 – 1
3 – 3
5 – 9
7 – $(\sqrt{2}+1)^2(1+2\sqrt{2})$
9 – 49
11 – 99
15 – $3(5+4\sqrt{2})^2$
23 – $9(\sqrt{2}+1)^4(3+4\sqrt{2})$
29 – 99^2
35 – $63(8\sqrt{2}+5\sqrt{5})^2$
71 – $9(\sqrt{2}+1)^{10}(2\sqrt{2}+1)^2(6\sqrt{2}+1)$.

$$F\,\frac{1-\sqrt{1-\frac{1}{4}\left(\frac{\sqrt{6}+3\sqrt{3}-\sqrt{2}+3\sqrt{3}}{2}\right)^{18}(\sqrt{3}+1)^6(2\sqrt{3}\sqrt{3}+2-3\sqrt{3}\sqrt{3}-2)^2}}{2}$$

$$= e^{-\pi\sqrt{69}}.$$

$$F\,\frac{1-\sqrt{1-4\left(\frac{\sqrt{6}+3\sqrt{3}-\sqrt{2}+3\sqrt{3}}{2}\right)^{16}\left(\frac{\sqrt{3}+1}{2}\right)^6(2\sqrt{3}\sqrt{3}+2+3\sqrt{3}\sqrt{3}-2)^2}}{2}$$

$$= e^{-\pi\sqrt{\frac{23}{3}}}.$$

$\sqrt{49}$ $(8-3\sqrt{7})^2(7-4\sqrt{3})^2(3-2\sqrt{2})^2(\sqrt{7}-\sqrt{6})^2$.

$\sqrt{72}$ $(2-\sqrt{3})^6(\sqrt{13}-2\sqrt{3})^4(3\sqrt{3}-\sqrt{26})^2(5-2\sqrt{6})^2$.

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$$F\,\frac{1-\sqrt{1-\left(\sqrt{\frac{2+\sqrt{73}}{8}}-\sqrt{\frac{1+\sqrt{73}}{8}}\right)^{24}}}{2} = e^{-\pi\sqrt{73}}$$

~~$$F\,\frac{1-\sqrt{1-\left(\sqrt{\frac{15+\sqrt{193}}{4}}-\sqrt{\frac{11+\sqrt{193}}{4}}\right)^{24}}}{2} = e^{-\pi\sqrt{193}}$$~~

$$F\,\frac{1-\sqrt{1-\left(\sqrt[3]{\frac{1+\sqrt{\frac{43}{27}}}{2}}+\sqrt[3]{\frac{1-\sqrt{\frac{43}{27}}}{2}}\right)^{24}}}{2} = e^{-\pi\sqrt{43}}$$

$$1+\frac{x^{\frac{1}{5}}}{1-x^{\frac{1}{5}}}+\frac{2x^{\frac{2}{5}}}{1-x^{\frac{2}{5}}}+\&c$$

$$-25\left(\frac{x^5}{1-x^5}+\frac{2x^{10}}{1-x^{10}}+\&c\right)$$

$$= \frac{f(-x)}{f(-x^5)}\sqrt{1+2x^{\frac{1}{5}}\frac{f(-x^5)}{f(-x^{\frac{1}{5}})}+5x^{\frac{2}{5}}\frac{f^2(-x^5)}{f^2(-x^{\frac{2}{5}})}}$$

$$\sqrt{102}\left(\frac{\sqrt{51}-7}{\sqrt{2}}\right)^4(5-2\sqrt{6})^4(\sqrt{51}-5\sqrt{2})^2(2-\sqrt{3})^4.$$

$$\sqrt{130}\,(5\sqrt{130}-57)^2(3-2\sqrt{2})^4(\sqrt{26}-5)^4(\sqrt{10}-3)^4.$$

$$\sqrt{190}\left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^4(37\sqrt{19}-51\sqrt{10})^2(2\sqrt{5}-\sqrt{19})^4(\sqrt{19}-3\sqrt{2})^4$$

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$F\left(\frac{1-\sqrt{1-\left(\sqrt{\frac{6+3\sqrt{3}}{4}}-\sqrt{\frac{2+3\sqrt{3}}{4}}\right)^{12}}{2}\right)=e^{-\pi\sqrt{69}}$

where $C=\left(\sqrt{\frac{(5+3\sqrt{3})\sqrt{4+6\sqrt{3}}+1}{2}}\pm\sqrt{\frac{(5+3\sqrt{3})\sqrt{4+6\sqrt{3}}-1}{2}}\right)^4$

$4\beta(1-\beta)=\left(\frac{\sqrt{13}-3}{2}\right)^6(\sqrt{13}-\sqrt{12})^4\left(\sqrt{\frac{11+6\sqrt{3}}{2}}\pm\sqrt{\frac{9+6\sqrt{3}}{2}}\right)^8$ for $e^{-3\pi\sqrt{13}}$

$4\beta(1-\beta)=(\sqrt{37}-6)^6(2\sqrt{37}-7\sqrt{3})^4\left(\sqrt{73+42\sqrt{3}}\pm\sqrt{72+42\sqrt{3}}\right)^8$ for $e^{-3\pi\sqrt{37}}$.

$4\beta(1-\beta)=\left(\frac{\sqrt[3]{2(\sqrt{3}-1)}-1}{\sqrt[3]{2(\sqrt{3}+1)}+1}\right)^8$ for $e^{-9\pi}$.

$4\beta(1-\beta)=\frac{1}{4}\cdot\left\{\frac{1\pm\left(2\sqrt[6]{\frac{28}{27}}-\sqrt{\frac{7}{3}}\right)}{2}\right\}^{24}$ for $e^{-7\pi\sqrt{3}}$.

$4\beta(1-\beta)=\frac{x^{24}}{2^{10}}$ for $e^{-11\pi\sqrt{3}}$ where $=0$

$x^8+x^4(1\pm\sqrt{11+2\sqrt{33}})+x(4+\sqrt{33}\pm\sqrt{11+2\sqrt{33}})-2$

$\sqrt{217}\left(\sqrt{\frac{11+4\sqrt{7}}{2}}-\sqrt{\frac{9+4\sqrt{7}}{2}}\right)^{12}\left(\sqrt{\frac{16+5\sqrt{7}}{4}}\pm\sqrt{\frac{12+5\sqrt{7}}{4}}\right)^{12}$

$\sqrt{205}\left(\sqrt{5}-2\right)^8\left(\frac{3\sqrt{5}-\sqrt{41}}{2}\right)^6\left(\sqrt{\frac{7+\sqrt{41}}{8}}\pm\sqrt{\frac{\sqrt{41}-1}{8}}\right)^{24}$

$\sqrt{265}\left(\frac{\sqrt{53}\pm1}{2}\right)^6(15\pm2)^6\left(\sqrt{\frac{89+5\sqrt{265}}{8}}-\sqrt{\frac{81+5\sqrt{265}}{8}}\right)^{12}$

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$\sqrt{57}$. $\frac{1}{4}(3\sqrt{19}-13)^4(2\pm\sqrt{3})^6$

$\sqrt{93}$. $\frac{1}{2}(39-7\sqrt{31})^4\left(\frac{\sqrt{31}\pm 3\sqrt{3}}{2}\right)^6$ 12

$\sqrt{177}$ $\frac{1}{4}(3\sqrt{59}\pm 23)^4(2-\sqrt{3})^{18}$.

$\sqrt{85}$ $(\sqrt{5}\pm 2)^8\left(\frac{\sqrt{85}-9}{2}\right)^6$.

$\sqrt{133}$ $(8-3\sqrt{7})^6\left(\frac{5\sqrt{7}\pm 3\sqrt{19}}{2}\right)^6$.

$\sqrt{55}$ $\frac{1}{64}(\sqrt{5}-2)^4\left(\sqrt{\frac{7+\sqrt{5}}{8}}\pm\sqrt{\frac{\sqrt{5}-1}{8}}\right)^{24}$ 12

$\sqrt{65}$ $\left(\frac{\sqrt{13}\pm 3}{2}\right)^6(\sqrt{5}\pm 2)^2\left(\sqrt{\frac{9+\sqrt{65}}{8}}-\sqrt{\frac{1+\sqrt{65}}{8}}\right)^{12}$.

$\sqrt{253}$ $(24-5\sqrt{23})^6\left(\frac{9\sqrt{23}\pm 13\sqrt{11}}{2}\right)^6$.

$\sqrt{145}$ $(\sqrt{5}-2)^6\left(\frac{\sqrt{29}-5}{2}\right)^6\left(\sqrt{\frac{17+\sqrt{145}}{8}}\pm\sqrt{\frac{9+\sqrt{145}}{8}}\right)^{12}$.

$\sqrt{117}$. $\left(\frac{\sqrt{13}-3}{2}\right)^6(\sqrt{13}-2\sqrt{3})^4\left(\frac{\sqrt{4+\sqrt{3}}\pm\sqrt{3}}{2}\right)^{24}$.

$\sqrt{333}$ $(\sqrt{37}-6)^6(2\sqrt{37}-7\sqrt{3})^4\left(\frac{\sqrt{7+2\sqrt{3}}\pm\sqrt{3+2\sqrt{3}}}{2}\right)^{24}$.

$\sqrt{153}$ $\left(\sqrt{\frac{5+\sqrt{17}}{8}}-\sqrt{\frac{\sqrt{17}-3}{8}}\right)^{48}\left(\sqrt{\frac{37+9\sqrt{17}}{4}}\pm\sqrt{\frac{33+9\sqrt{17}}{4}}\right)^8$

$\sqrt{77}$ $(8\pm 3\sqrt{7})^3\left(\frac{\sqrt{11}\pm\sqrt{7}}{2}\right)^3\left(\sqrt{\frac{6+\sqrt{11}}{4}}-\sqrt{\frac{2\pm\sqrt{11}}{4}}\right)^{12}$.

$\sqrt{69}$ $\left(\frac{5\pm\sqrt{23}}{\sqrt{2}}\right)^2\left(\frac{3\sqrt{3}\pm\sqrt{23}}{2}\right)^3\left(\sqrt{\frac{6+3\sqrt{3}}{4}}-\sqrt{\frac{2\pm 3\sqrt{3}}{4}}\right)^{12}$

$\sqrt{213}$ $\left(\frac{59\pm 7\sqrt{71}}{2}\right)^2\left(\frac{5\sqrt{3}\pm\sqrt{71}}{2}\right)^3\left(\sqrt{\frac{21+12\sqrt{3}}{2}}-\sqrt{\frac{19\pm 12\sqrt{3}}{2}}\right)^{12}$

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$$(1-e^{-\pi\sqrt{n}})(1-e^{-3\pi\sqrt{n}})(1-e^{-5\pi\sqrt{n}})\&c$$
$$= \sqrt[4]{2}\cdot e^{-\frac{\pi}{24}\sqrt{n}}\cdot g_n$$

$g_2 = 1$; $g_6 = \sqrt[6]{1+\sqrt{2}}$; $g_{10} = \sqrt{\frac{1+\sqrt{5}}{2}}$; $g_{14} = \sqrt{\frac{3+\sqrt{2}}{4}} + \sqrt{\frac{\sqrt{2}-1}{4}}$

$g_{18} = \sqrt[3]{\sqrt{2}+\sqrt{3}}$; $g_{22} = \sqrt{1+\sqrt{2}}$; $g_{30} = \sqrt{\frac{1+\sqrt{5}}{2}}\sqrt[6]{3+\sqrt{10}}$.

$g_{37} = \sqrt{\frac{5+\sqrt{29}}{2}}$; $g_{70} = \frac{1+\sqrt{5}}{2}\sqrt{1+\sqrt{2}}$. $g_{46} = \sqrt{\frac{5+\sqrt{2}}{4}} + \sqrt{\frac{\sqrt{2}-1}{4}}$

$g_{142} = \sqrt{\frac{11+5\sqrt{2}}{4}} + \sqrt{\frac{7+5\sqrt{2}}{4}}$. $g_{82} = \sqrt{\frac{13+\sqrt{41}}{8}} + \sqrt{\frac{5+\sqrt{41}}{8}}$

$$\sqrt{17.5}\cdot\left\{\frac{\sqrt{5}-1}{2} + \sqrt[3]{\frac{5-\sqrt{5}}{4}}\left(\sqrt[3]{8-3\sqrt{5}+3\sqrt{21}} + \sqrt[3]{8-3\sqrt{5}-3\sqrt{21}}\right)\right\}$$
$$64.3^{24}$$

$$\sqrt{\frac{25}{7}}\cdot\left\{-\frac{\sqrt{5}+1}{2} + \sqrt[3]{\frac{5+\sqrt{5}}{4}}\left(\sqrt[3]{8+3\sqrt{5}+3\sqrt{21}} + \sqrt[3]{8+3\sqrt{5}-3\sqrt{21}}\right)\right\}$$
$$64.3^{24}$$

$$\sqrt{\frac{17+\sqrt{17}+\sqrt{170+42\sqrt{17}}}{16}}$$

$$\sqrt{\frac{1+\sqrt{17}+\sqrt{170+42\sqrt{17}}}{8}}$$

$g_{58} = \sqrt{\frac{17+\sqrt{13}}{8}}\sqrt[6]{2\sqrt{2}+\sqrt{7}}$. $g_{78} = \sqrt{\frac{3+\sqrt{13}}{2}}\sqrt[6]{5+\sqrt{26}}$

$g_{102} = \sqrt{1+\sqrt{2}}\sqrt[3]{3\sqrt{2}+\sqrt{17}}$. $g_{130} = \sqrt{\frac{3+\sqrt{13}}{2}}\sqrt{2+\sqrt{5}}$

$\sqrt{2+\sqrt{5}}\sqrt{3+\sqrt{10}}$ $g_{34} = \sqrt{\frac{7+\sqrt{17}}{8}} + \sqrt{\frac{\sqrt{17}-1}{8}}$

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$\sqrt{289}\cdot\left\{\sqrt{\frac{17+\sqrt{17}}{16}+\frac{(5+\sqrt{17})\sqrt[4]{17}}{16}}-\sqrt{\frac{1+\sqrt{17}}{16}+\frac{(5+\sqrt{17})\sqrt[4]{17}}{16}}\right\}^{48}$

$\sqrt{121}\cdot\left\{\frac{\sqrt[3]{11-3\sqrt{11}}\left(\sqrt[3]{3\sqrt{11}+3\sqrt{3}-4}+\sqrt[3]{3\sqrt{11}-3\sqrt{3}-4}\right)-2}{3\sqrt{2}}\right\}^{24}$

$\sqrt{169}\cdot\left\{\frac{\sqrt[3]{\frac{13-3\sqrt{13}}{2}}\left(\sqrt[3]{3\sqrt{3}-\frac{11-\sqrt{13}}{2}}-\sqrt[3]{3\sqrt{3}+\frac{11-\sqrt{13}}{2}}\right)+(\sqrt{13}-2)}{3}\right\}^{24}$

$\sqrt{105}\cdot\left(\frac{5-\sqrt{21}}{2}\right)^{6}(2\pm\sqrt{3})^{6}(\sqrt{5}\pm 2)^{4}(6\pm\sqrt{35})^{2}$.

$\sqrt{165}\cdot(6-\sqrt{15})^{6}(3\sqrt{5}\pm 2\sqrt{11})^{4}\left(\frac{\sqrt{15}\pm\sqrt{11}}{2}\right)^{6}(\sqrt{5}\pm 2)^{4}$.

$\sqrt{345}\cdot\left(\frac{3\sqrt{3}-\sqrt{23}}{2}\right)^{12}\left(\frac{7\sqrt{23}\pm 15\sqrt{5}}{\sqrt{2}}\right)^{4}(\sqrt{5}\pm 2)^{8}(2\pm\sqrt{3})^{6}$

$\sqrt{385}\cdot(10-3\sqrt{11})^{6}(5\pm\sqrt{35})^{6}\left(\frac{\sqrt{11}\pm\sqrt{7}}{2}\right)^{12}(\sqrt{5}\pm 2)^{8}$.

$\sqrt{273}\cdot\left(\frac{15\sqrt{7}-11\sqrt{13}}{\sqrt{2}}\right)^{4}\left(\frac{\sqrt{13}\pm 3}{2}\right)^{12}\left(\frac{\sqrt{7}\pm\sqrt{3}}{2}\right)^{12}(2\pm\sqrt{3})^{6}$.

$\sqrt{357}\cdot\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{24}(8\pm 3\sqrt{7})^{6}\left(\frac{11\pm\sqrt{119}}{\sqrt{2}}\right)^{4}\left(\frac{\sqrt{21}\pm\sqrt{17}}{2}\right)^{6}$.

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$$g_{98} = \left(\sqrt{\frac{\sqrt{2}\cdot\sqrt{14}+\sqrt{14}+4\sqrt{14}}{8}} + \sqrt{\frac{\sqrt{2}\cdot\sqrt{14}-4+\sqrt{14}+4\sqrt{14}}{8}} \right).$$

$$g_{90} = \sqrt{\frac{\sqrt{5}+1}{2}}\ \sqrt[6]{\sqrt{6}+\sqrt{5}}\left(\sqrt{\frac{3+\sqrt{6}}{4}} + \sqrt{\frac{\sqrt{6}-1}{4}} \right).$$

$$g_{198} = \sqrt{\sqrt{2}+1}\ \sqrt[6]{4\sqrt{2}+\sqrt{33}}\left(\sqrt{\frac{9+\sqrt{33}}{8}} + \sqrt{\frac{1+\sqrt{33}}{8}} \right).$$

If $g_n^4 - \frac{1}{g_n^4} = p$, then $g_{9n} = g_n \sqrt[6]{p+\sqrt{p^2+1}} \times$

$$\sqrt[3]{\left\{ \sqrt{\frac{p^2+4+\sqrt{(p^2+1)(p^2+4)}}{2}} + \sqrt{\frac{p^2+2+\sqrt{(p^2+1)(p^2+4)}}{2}} \right\}}.$$

$$g_{522} = \sqrt{\frac{5+\sqrt{29}}{2}}\ \sqrt[6]{5\sqrt{29}+11\sqrt{6}}\left(\sqrt{\frac{9+3\sqrt{6}}{4}} + \sqrt{\frac{5+3\sqrt{6}}{4}} \right)$$

$$g_{630} = \frac{1+\sqrt{5}}{2}\sqrt{1+\sqrt{2}}\ \sqrt[6]{\sqrt{15}+\sqrt{14}}\ \sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}}\left(\sqrt{\frac{2+\sqrt{7}+\sqrt{15}}{4}} + \sqrt{\frac{\sqrt{7}+\sqrt{15}-2}{4}} \right)$$
$$\times \left(\sqrt{\frac{4+\sqrt{7}+\sqrt{15}}{8}} + \sqrt{\frac{\sqrt{7}+\sqrt{15}-4}{8}} \right).$$

$$g_{1170} = \sqrt{2}+\sqrt{5}\ \sqrt{\frac{3+\sqrt{13}}{2}}\ \sqrt{\sqrt{2}+\sqrt{3}}\ \sqrt[6]{2\sqrt{10}+\sqrt{39}}$$

$$g_{50} = \frac{1+\sqrt[3]{\frac{5+\sqrt{5}}{4}}\left(\sqrt[3]{1+7\sqrt{5}+6\sqrt{6}} + \sqrt[3]{1+7\sqrt{5}-6\sqrt{6}} \right)}{3}.$$

$$g_{126} = \sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}}\ \sqrt[6]{\sqrt{7}+\sqrt{6}}\left(\sqrt{\frac{3+\sqrt{2}}{4}} + \sqrt{\frac{\sqrt{2}-1}{4}} \right)^2.$$

$$g_{26} = \frac{1}{3}\left\{ \sqrt{2+\sqrt{13}} + \sqrt[3]{(2+\sqrt{13})\sqrt{2+\sqrt{13}}} + \sqrt[3]{3(3+\sqrt{13})} \right.$$
$$\left. \text{etc.} - \sqrt[3]{(2+\sqrt{13})\sqrt{2+\sqrt{13}}} - \sqrt[3]{3(3+\sqrt{13})} \right\}$$

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$$g_{66} = \sqrt[4]{\sqrt{2}+\sqrt{3}}\,\sqrt[12]{3\sqrt{11}+7\sqrt{2}}\,\sqrt{\sqrt{\frac{7+\sqrt{33}}{8}} \pm \sqrt{\frac{\sqrt{33}-1}{8}}}$$

$$g_{138} = \sqrt[4]{\frac{3\sqrt{3}+\sqrt{23}}{2}}\,\sqrt[12]{23\sqrt{23}+78\sqrt{2}}\,\sqrt{\sqrt{\frac{5+2\sqrt{6}}{4}} \pm \sqrt{\frac{1\pm 2\sqrt{6}}{4}}}$$

$$g_{154} = \sqrt[4]{2\sqrt{2}\mp\sqrt{7}}\,\sqrt[4]{\frac{\sqrt{11}\pm\sqrt{7}}{2}}\,\sqrt{\sqrt{\frac{9+2\sqrt{22}}{4}} \mp \sqrt{\frac{5+2\sqrt{22}}{4}}}$$

$$g_{114} = \sqrt[4]{\sqrt{3}\pm\sqrt{2}}\,\sqrt[12]{\sqrt{19}\pm 3\sqrt{2}}\,\sqrt{\frac{3\sqrt{3}+\sqrt{19}}{4}} + \sqrt{\frac{15+3\sqrt{57}}{8}}$$

$$g_{238} = \left(\sqrt{\frac{5+3\sqrt{2}}{4}} + \sqrt{\frac{1+3\sqrt{2}}{4}}\right)\left(\sqrt{\frac{5+2\sqrt{2}}{4}} \pm \sqrt{\frac{1+2\sqrt{2}}{4}}\right)$$

$$g_{62} = \left(\sqrt{\frac{4+\sqrt{1+\sqrt{2}}+\sqrt{9+5\sqrt{2}}}{8}} + \sqrt{\frac{\sqrt{1+\sqrt{2}}+\sqrt{9+5\sqrt{2}}-4}{8}}\right)$$

$$g_{94} = \left(\sqrt{\frac{4+\sqrt{7+\sqrt{2}}+\sqrt{7+5\sqrt{2}}}{8}} + \sqrt{\frac{\sqrt{7+\sqrt{2}}+\sqrt{7+5\sqrt{2}}-4}{8}}\right)$$

$$g_{154} = \sqrt[4]{2\sqrt{2}\pm\sqrt{7}}\,\sqrt[4]{\frac{\sqrt{11}\pm\sqrt{7}}{2}}\,\sqrt{\sqrt{\frac{13+2\sqrt{22}}{4}} + \sqrt{\frac{9+2\sqrt{22}}{4}}}$$

$$g_{310} = \sqrt{\frac{5}{2}\pm\frac{1}{2}}\,\sqrt{\sqrt{2}\pm 1}\left(\sqrt{\frac{7+2\sqrt{10}}{4}} + \sqrt{\frac{3+2\sqrt{10}}{4}}\right)$$

$$g_{158} = \left(\sqrt{\frac{4+\sqrt{9+\sqrt{2}}+\sqrt{17+13\sqrt{2}}}{8}} + \sqrt{\frac{\sqrt{9+\sqrt{2}}+\sqrt{17+13\sqrt{2}}-4}{8}}\right)$$

$$\sqrt{465}\left(\sqrt{\frac{13+2\sqrt{31}}{2}} - \sqrt{\frac{11+2\sqrt{31}}{2}}\right)^{12}\left(\frac{\sqrt{31}-3\sqrt{3}}{2}\right)^{6}(2-\sqrt{3})^{6}$$
$$\times\left(\sqrt{\frac{6+\sqrt{31}}{4}} - \sqrt{\frac{2+\sqrt{31}}{4}}\right)^{12}(5\sqrt{5}-2\sqrt{31})^{2}(\sqrt{5}-2)^{2}$$

$$\sqrt{777}\left(\sqrt{37}-6\right)^{6}(107\sqrt{37}-246\sqrt{7})^{2}\left(\sqrt{\frac{17+6\sqrt{7}}{2}} - \sqrt{\frac{15+6\sqrt{7}}{2}}\right)^{12}$$
$$\times\left(\sqrt{\frac{10+3\sqrt{7}}{4}} - \sqrt{\frac{6+3\sqrt{7}}{4}}\right)^{12}(2-\sqrt{3})^{6}\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{6}$$

$$\sqrt{1353}\left(\sqrt{\frac{569+99\sqrt{33}}{8}} - \sqrt{\frac{561+99\sqrt{33}}{8}}\right)^{12}\left(\frac{321\sqrt{45}-6817}{\sqrt{2}}\right)^{2}$$
$$\times\left(\frac{\sqrt{133}-11}{\sqrt{2}}\right)^{6}(10-3\sqrt{11})^{3}(2-\sqrt{3})^{9}\left(\sqrt{\frac{25+3\sqrt{33}}{8}} - \sqrt{\frac{17+3\sqrt{33}}{8}}\right)^{12}$$

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$\sqrt{1645}$ $\left(\sqrt{\frac{751+41\sqrt{329}}{8}} - \sqrt{\frac{743+41\sqrt{329}}{8}}\right)^{12}\left(\frac{9\sqrt{329}-73\sqrt{5}}{2}\right)^{3}$

$\times(\sqrt{5}-2)^{12}\left(\sqrt{\frac{127+7\sqrt{329}}{8}} - \sqrt{\frac{119+7\sqrt{329}}{8}}\right)^{12}\left(\frac{7-\sqrt{47}}{\sqrt{2}}\right)^{6}(8-3\sqrt{7})^{3}$

$\sqrt{897}$ $q+\frac{1}{q} = 58+9\sqrt{39}$ $s+\frac{1}{s} = 6+\sqrt{39}$.

$\sqrt{1677}$ $s+\frac{1}{s} = 15+2\sqrt{43}$

$\sqrt{141}$ $(4\sqrt{3}\pm\sqrt{47})^{3}\left(\frac{7\pm\sqrt{47}}{\sqrt{2}}\right)^{2}\left(\sqrt{\frac{18+9\sqrt{3}}{4}} - \sqrt{\frac{14+9\sqrt{3}}{4}}\right)^{12}$.

$\sqrt{445}$ $(\sqrt{5}-2)^{12}\left(\frac{\sqrt{445}-21}{2}\right)^{6}\left(\sqrt{\frac{13+\sqrt{89}}{8}} \pm \sqrt{\frac{5+\sqrt{89}}{8}}\right)^{24}$.

$\sqrt{553}$ $\left(\sqrt{\frac{143+16\sqrt{79}}{2}} - \sqrt{\frac{141+16\sqrt{79}}{2}}\right)^{12}\left(\sqrt{\frac{100+11\sqrt{79}}{4}} \pm \sqrt{\frac{96+11\sqrt{79}}{4}}\right)^{12}$

$g_{210} = \sqrt{\sqrt{3}+\sqrt{2}}\ \sqrt[6]{3\sqrt{14}+5\sqrt{5}}\ \sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}}\ \sqrt{\frac{\sqrt{5}+1}{2}}$

$g_{330} = \sqrt{\sqrt{6}+\sqrt{5}}\ \sqrt{\frac{\sqrt{15}+\sqrt{11}}{2}}\cdot\frac{\sqrt{5}+1}{2}\cdot\sqrt[6]{\sqrt{11}+\sqrt{10}}$.

~~If u & v are~~ If $g_n^6 = uv$; $u^2+\frac{1}{u^2} = 2U$, $v^2+\frac{1}{v^2} = 2V$,

$\sqrt{U^2+V^2-1} = W$ and $2S = U+V+W+1$, then

$F\ (\sqrt{S}-\sqrt{S-1})^2(\sqrt{S-U}-\sqrt{S-U-1})^2(\sqrt{S-V}-\sqrt{S-V-1})^2(\sqrt{S-W}-\sqrt{S-W-1})^2$

$= e^{-\pi\sqrt{n}}$.

$\sqrt{210}$ $(4-\sqrt{15})^4(\sqrt{10}-3)^4(\sqrt{7}-\sqrt{6})^4(8-3\sqrt{7})^2(6-\sqrt{35})^2$

$\times(\sqrt{15}-\sqrt{14})^2(3-2\sqrt{2})^2(2-\sqrt{3})^2$.

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If $2u = 11 + \dfrac{f^6(-x)}{x f^6(-x^5)}$ and $2v = 1 + \dfrac{f(-x^{\frac{1}{5}})}{x^{\frac{1}{5}} - f(-x^5)}$

then $\sqrt[5]{\sqrt{u^2+1} - u} = \sqrt{v^2+1} - v = \cfrac{\sqrt[5]{x}}{1+\cfrac{x}{1+\cfrac{x^2}{1+\cfrac{x^3}{1+\&c}}}}$

$$3 + \frac{f^3(-x^{\frac{1}{3}})}{x^{\frac{1}{3}} f^3(-x^3)} = \sqrt[3]{27 + \frac{f^{12}(-x)}{x f^{12}(-x^3)}}$$

$$f^3(-x^{\frac{1}{3}}) + 3x^{\frac{1}{3}} f^3(-x^3) = f(-x)\left\{1 + 6\left(\frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} - \&c\right)\right\}$$

$$1 + \frac{\psi(-x^{\frac{1}{3}})}{x^{\frac{1}{3}}\psi(-x^3)} = \sqrt[3]{1 + \frac{\psi^4(-x)}{x\psi^4(-x^3)}}$$

$$\frac{\phi(x^{\frac{1}{3}})}{\phi(x^3)} - 1 = \sqrt[3]{\frac{\phi^4(x)}{\phi^4(x^3)} - 1}$$

$$1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.1}{2.4}\right)^2 x^2 + \left(\frac{1.1.3}{2.4.6}\right)^2 x^3 + \&c$$

$$= \frac{7}{3}(1+x) + \frac{2}{3\pi}\left\{1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)\right\}$$

$$1 - \frac{1^2}{2^2}x - \frac{1^2.3}{2^2.4^2}x^2 - \frac{1^2.3^2.5}{2^2.4^2.6^2}x^3 - \&c$$

$$= \frac{7}{3}(2-x) + \frac{1}{3\pi}\left\{1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)\right\}$$

$$\frac{\cos\theta + 2\cos\frac{\theta}{2}\cosh\frac{\theta}{2}\sqrt{3}}{\cosh\frac{\pi\sqrt{3}}{2}} - \frac{\cos 3\theta + 2\cos\frac{3\theta}{2}\cosh\frac{3\theta\sqrt{3}}{2}}{3\cosh\frac{3\pi\sqrt{3}}{2}} + \&c$$

$$= \frac{\pi}{8}.$$

$$\frac{1^5}{16 - x^6}\cdot\frac{1}{\cosh\frac{\pi\sqrt{3}}{2}} - \frac{3^5}{36 - x^6}\cdot\frac{1}{\cosh\frac{3\pi\sqrt{3}}{2}} + \&c$$

$$= \frac{\pi}{12}\cdot\frac{1}{\cos\frac{\pi x}{2}\left\{\cos\frac{\pi x}{2} + \cosh\frac{\pi x\sqrt{3}}{2}\right\}}$$

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If $u = \frac{f^4(-x)}{x f^4(-x^7)}$ and $v = \frac{f(-x^{\frac{1}{7}})}{x^{\frac{2}{7}} f(-x^7)}$, then

$$2u = 7(v^3 + 5v^2 + 7v) + (v^2 + 7v + 7)\sqrt{4v^3 + 21v^2 + 28v}.$$

$$1 + 12\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \&c\right) - 12\left(\frac{3x^3}{1-x^3} + \frac{6x^6}{1-x^6} + \&c\right)$$
$$= \frac{\{\psi^4(x) + 3x\psi^4(x^3)\}^2}{\psi^2(x)\psi^2(x^3)} = \frac{\{f^{12}(-x) + 27x f^{12}(-x^3)\}^{\frac{2}{3}}}{f^2(-x) f^2(-x^3)}$$
$$= \frac{\left\{\frac{\phi^4(\sqrt{x}) + 3\phi^4(\sqrt{x^3})}{4\phi(\sqrt{x})\phi(\sqrt{x^3})}\right\}^2}{}$$

$$1 + 12\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \&c\right) - 12\left(\frac{3x^6}{1-x^6} + \frac{6x^{12}}{1-x^{12}} + \&c\right)$$
$$= \left\{\frac{\phi^4(x) + 3\phi^4(x^3)}{4\phi(x)\phi(x^3)}\right\}^2 = \phi^2(x)\phi^2(x^3)\left\{1 - \frac{4x}{\chi^6(x)\chi^6(x^3)}\right\}$$

$$1 + 6\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \&c\right) - 6\left(\frac{5x^5}{1-x^5} + \frac{10x^{10}}{1-x^{10}} + \&c\right)$$
$$= \sqrt{f^{12}(-x) + 22x f^6(-x) f^6(-x^5) + 125x^2 f^{12}(-x^5)} \Big/ f(-x) f(-x^5).$$
$$= \left\{\psi^4(x) + 20\psi^2(x)\psi^2(x^5) + 5x^2\psi^4(x^5)\right\}\sqrt{\psi^4(x) - 2x\psi^2(x)\psi^2(x^5) + 5x^2\psi^4(x^5)}$$
$$\div \psi(x)\psi(x^5).$$

$$1 + 6\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \&c\right) - 6\left(\frac{5x^{10}}{1-x^{10}} + \frac{10x^{20}}{1-x^{20}} + \&c\right)$$
$$= \phi^2(x)\phi^2(x^5)\left\{1 - \frac{2x}{\chi^4(x)\chi^4(x^5)}\right\}\sqrt{1 - \frac{4x}{\chi^4(x)\chi^4(x^5)}}.$$

$$1 + 4\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right) - 4\left(\frac{7x^7}{1-x^7} + \frac{14x^{14}}{1-x^{14}} + \&c\right)$$
$$= \left\{\frac{f^8(-x) + 13x f^4(-x) f^4(-x^7) + 49x^2 f^8(-x^7)}{f(-x) f(-x^7)}\right\}^{\frac{2}{3}}$$

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$$1+4\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\right)-4\left(\frac{7x^{14}}{1-x^{14}}+\frac{14x^{28}}{1-x^{28}}+\&c\right)$$

$$=\phi^2(x)\,\phi^2(x^7)\left\{1-\frac{2x}{x^3(x)\,x^3(x^7)}\right\}^2.$$

$$1+3\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\&c\right)-3\left(\frac{9x^9}{1-x^9}+\frac{18x^{18}}{1-x^{18}}+\&c\right)$$

$$=\frac{c^6(-x^3)}{f^2(-x)\,f^2(-x^9)}\left\{f^6(-x)+9x\,f^3(-x)\,f^3(-x^9)+27x^2f^6(-x^9)\right\}^{\frac{1}{3}}$$

$$=\left\{\frac{\phi^4(x^3)+3\,\phi^2(x)\,\phi^2(x^9)}{4}\right\}^2\cdot\frac{\phi^2(x^3)}{\phi^3(x)\,\phi^3(x^9)}$$

$$1+3\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-3\left(\frac{9x^{18}}{1-x^{18}}+\&c\right)$$

$$*\quad\frac{f(-x^2,-x^7)}{f(-x,-x^8)}=\frac{f(-x^4,-x^5)}{f(-x^2,-x^7)}+x\,\frac{f(-x,-x^8)}{f(-x^4,-x^5)}$$

$$f(-x^5)=f(-x^4,-x^5)-x\tfrac{1}{3}f(-x^2,-x^7)-x^{\frac{2}{3}}f(-x,-x^8).$$

$$*\quad-\frac{1}{u}+\frac{1}{v}+\frac{1}{w}=\frac{f^4(-x^3)}{f(-x)\,f^3(-x^3)}$$

$$5+12\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\frac{3x^6}{1-x^6}+\&c\right)$$

$$-12\left(\frac{11x^{22}}{1-x^{22}}+\frac{22x^{44}}{1-x^{44}}+\&c\right)$$

$$=5\,\phi^2(x)\,\phi^2(x^{11})-20x\,f^2(-x)\,f^2(-x^{11})+32x^2f^2(-x^2)\,f^2(-x^{22})$$

$$-20x^3\psi^2(-x)\,\psi^2(-x^{11}).$$

$$=\phi^2(x)\,\phi^2(x^{11})\left[5\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}-\frac{1}{2}\left\{1-\sqrt[4]{\alpha\beta}-\sqrt[4]{(1-\alpha)(1-\beta)}\right\}^2\right]$$

$$3+4\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-4\left(\frac{19x^{38}}{1-x^{38}}+\frac{38x^{76}}{1-x^{76}}+\&c\right)$$

$$=\phi^2(x)\,\phi^2(x^{19})\left[3\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}-\frac{1}{2}\left\{1-\sqrt[4]{\alpha\beta}\cdot\sqrt[4]{(1-\alpha)(1-\beta)}\right\}^2\right]$$

$$11+12\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-12\left(\frac{23x^{46}}{1-x^{46}}+\frac{46x^{92}}{1-x^{92}}+\&c\right)$$

$$=\phi^2(x)\,\phi^2(x^{23})\left\{11\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}\right.$$

$$-16\sqrt[3]{2}\cdot\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}\cdot\frac{1+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}}{2}$$

$$\left.-10\sqrt[3]{4}\cdot\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}\right\}$$

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$$\frac{1}{2}+12\left(\frac{x^2}{1-x^2}+\frac{x^4}{1-x^4}+\&c\right)-12\left(\frac{15x^{30}}{1-x^{30}}+\frac{30x^{60}}{1-x^{60}}+\&c\right)$$

$$=\phi^2(x)\,\phi^2(x^{15})\left\{7\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}\right.$$

$$\left.-2\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\left\{1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\right\}\right.$$

$$\left.=\frac{1}{2}+\frac{2+1}{2+1}-\frac{8\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}}{\left(1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\right)^2}\right\}$$

$$5+4\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-4\left(\frac{31x^{62}}{1-x^{62}}+\frac{62x^{124}}{1-x^{124}}+\&c\right)$$

$$=\phi^2(x)\,\phi^2(x^{31})\left\{8\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}-6\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\left(1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\right)\right.$$

$$\left.-4\sqrt[16]{\alpha\beta(1-\alpha)(1-\beta)}\sqrt{1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}}\left(1+\sqrt[4]{\alpha\beta}+\sqrt[4]{1-\alpha}\sqrt[4]{1-\beta}\right)\right.$$

$$1+6\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-6\left(\frac{5x^{10}}{1-x^{10}}+\frac{10x^{20}}{1-x^{20}}+\&c\right)$$

$$=\phi^2(x)\,\phi^2(x^5)\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{1-\alpha}\sqrt{1-\beta}}{2}}\left\{1+\frac{\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}\right.$$

$$\left.+\frac{1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}}{4}\right.$$

$$=\phi^2(x)\,\phi^2(x^5)\sqrt{\frac{1+\alpha\beta+(1-\alpha)(1-\beta)}{2}}-\frac{3}{\sqrt[8]{4}}\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$$

$$v=\frac{\sqrt[3]{x}}{1+\dfrac{x+x^2}{1+\dfrac{x^2+x^4}{1+\dfrac{x^3+x^6}{1+\&c}}}}=\sqrt[3]{x}\,\frac{\chi(-x)}{\chi^3(-x^3)}$$

$$\frac{1}{v}+1=\frac{\psi(x^{\frac{1}{3}})}{x^{\frac{1}{3}}\psi(x^3)};\quad \frac{1}{v^3}+1=\frac{\psi^4(x)}{x\psi^4(x^3)}.$$

$$v=\frac{\sqrt{x}}{1+x+\dfrac{x^2}{1+x^3+\dfrac{x^4}{1+x^5+\dfrac{x^6}{1+x^7+\&c}}}}=\sqrt{x}\,\frac{f(-x,-x^7)}{f(-x^3,-x^5)}$$

$$\frac{1}{v}-v=\frac{\phi(x^2)}{\sqrt{x}\,\psi(x^4)};\quad \frac{1}{v}+v=\frac{\phi(x)}{\sqrt{x}\,\psi(x^4)}.$$

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/301 $(8\pm3\sqrt{7})^3\left(\frac{23\sqrt{43}\pm57\sqrt{7}}{2}\right)^3\times$

$$\left(\sqrt{\frac{46+7\sqrt{63}}{4}}-\sqrt{\frac{42+7\sqrt{63}}{4}}\right)^{12}.$$

$$f(-x^7,-x^8)+xf(-x^2,-x^{13}) = \frac{f(-x^2,-x^3)}{f(-x,-x^4)}f(-x^5).$$

$$f(-x^4,-x^{11})-xf(-x,-x^{14}) = \frac{f(-x,-x^6)}{f(-x^2,-x^3)}f(-x^5).$$

$$f(-x,-x^{14})f(-x^4,-x^{11})f(-x^6,-x^9)f(-x^5)$$
$$= f(-x,-x^{14})f^3(-x^{15}).$$

$$f(-x^7,-x^8)f(-x^2,-x^{13})f(-x^3,-x^{12})f(-x^5)$$
$$= f(-x^2,-x^3)f^3(-x^{15}).$$

$$f(-x^7,-x^8)-xf(-x^2,-x^{13}) = f(-x^{\frac{2}{3}},-x)+x^{\frac{3}{3}}f(-x^7,-x^{12})$$

$$f(-x^4,-x^{11})+xf(-x,-x^{14}) = \frac{f(-x^6,-x^9)-f(-x^{\frac{4}{3}},-x^{\frac{5}{3}})}{x^{\frac{1}{3}}}.$$

$$\frac{\sqrt[8]{x}}{1+\frac{1}{1+x}+\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}+\frac{x^4}{1+x^4}+\&c} = \sqrt[8]{x}\cdot\frac{\psi(x)}{\phi(x)}.$$

$$\frac{f(-x^{\frac{1}{7}})}{f(-x^7)} = \frac{f(-x^2,-x^5)}{f(-x,-x^6)}-x^{\frac{1}{7}}\frac{f(-x^3,-x^4)}{f(-x^2,-x^5)}-x^{\frac{2}{7}}$$
$$+x^{\frac{5}{7}}\frac{f(-x,-x^4)}{f(-x^3,-x^5)}.$$

$$\frac{f(-x^{\frac{1}{11}})}{f(-x^{11})} = \frac{f(-x^4,-x^7)}{f(-x^2,-x^9)}-x^{\frac{1}{11}}\frac{f(-x^2,-x^9)}{f(-x,-x^{10})}-x^{\frac{2}{11}}\frac{f(-x^5,-x^8)}{f(-x^3,-x^8)}$$
$$+x^{\frac{5}{11}}+x^{\frac{7}{11}}\frac{f(-x^3,-x^8)}{f(-x^4,-x^7)}-x^{\frac{15}{11}}\frac{f(-x,-x^{10})}{f(-x^5,-x^6)}.$$

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$f(-x^{\frac{1}{7}}) + x^{\frac{1}{7}} f(-x^7) = \sqrt[7]{u} - \sqrt[7]{v} + \sqrt[7]{w}$

$\sqrt[7]{uvw} = x^{\frac{6}{7}} f^3(-x^7).$

$$u - v + w = \frac{f^8(-x) + 14x f^4(-x) f^4(-x^7) + 56x^2 f^8(-x^7)}{f(-x^7)}$$

$$uv - uw + vw = f^2(-x^7)\{f^{12}(-x) + 19x f^8(-x) f^4(-x^7) + 196x^2 f^4(-x) f^8(-x^7) + 289x^3 f^{12}(-x^7)$$

$$\frac{f(-x^{\frac{1}{13}})}{x^{\frac{7}{13}} f(-x^{13})} = \frac{f(-x^2,-x^9)}{x^{\frac{7}{13}} f(-x^5,-x^{11})} - \frac{f(-x^6,-x^7)}{x^{\frac{6}{13}} f(-x^3,-x^{10})} - \frac{f(-x^4,-x^8)}{x^{\frac{5}{13}} f(-x^4,-x^9)}$$

$$+ \frac{f(-x^5,-x^8)}{x^{\frac{2}{13}} f(-x^4,-x^9)} + 1 - x^{\frac{5}{13}} \frac{f(-x^3,-x^{10})}{f(-x^5,-x^8)}$$

$$+ x^{\frac{15}{13}} \frac{f(-x,-x^{12})}{f(-x^6,-x^7)}.$$

$$= u_1 - u_2 - u_3 + u_4 + 1 - u_5 + u_6$$

$$u_1 u_2 - u_3 u_5 - u_4 u_6 = 1 + \frac{f^2(-x)}{x f^2(-x^{13})}.$$

$$u_1 u_2 \cdot u_3 u_5 - u_3 u_5 \cdot u_4 u_6 + u_4 u_6 \cdot u_1 u_2 = 4 + \frac{f^2(-x)}{x f^2(-x^{13})}$$

$$u_1 u_2 u_3 u_4 u_5 u_6 = 1.$$

$$u_2 u_3 u_4 - u_1 u_5 u_6 = 3 + \frac{f^2(-x)}{x f^2(-x^{13})}.$$

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$$\frac{f(-x^{\frac{1}{17}})}{x^{\frac{12}{17}} f(-x^{17})} = \frac{f(-x^6,-x^{11})}{x^{\frac{12}{17}} f(-x^3,-x^{14})} - \frac{1}{x^{\frac{11}{17}}} \frac{f(-x^4,-x^{13})}{f(-x^2,-x^{15})}$$

$$- \frac{1}{x^{\frac{10}{17}}} \cdot \frac{f(-x^8,-x^9)}{f(-x^4,-x^{13})} + \frac{1}{x^{\frac{7}{17}}} \frac{f(-x^2,-x^{15})}{f(-x,-x^{16})}$$

$$+ \frac{1}{x^{\frac{3}{17}}} \frac{f(-x^7,-x^{10})}{f(-x^1,-x^{12})} - 1 - x^{\frac{3}{17}} \frac{f(-x^5,-x^{12})}{f(-x^6,-x^{11})}$$

$$+ x^{\frac{14}{17}} \frac{f(-x^3,-x^{14})}{f(-x^7,-x^{10})} - x^{\frac{28}{17}} \frac{f(-x,-x^{16})}{f(-x^8,-x^9)}$$

$$= u_1 - u_2 - u_3 + u_4 + u_5 - 1 - u_6 + u_7 - u_8$$

$$\therefore u_1 u_5 + u_2 u_8 - u_3 u_4 - u_6 u_7 = -1$$

$$u_1 u_5 \cdot u_6 u_7 = 1$$

$$u_2 u_8 \cdot u_3 u_4 = 1$$

1, 3, 9, 27 $\quad \frac{\phi(x^3)\phi(x^9)}{\phi(x)\phi(x^{27})} = \frac{\sqrt[4]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[4]{\alpha\beta\gamma(1-\gamma)(1-\delta)}} + \frac{\sqrt[4]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[4]{\alpha\beta\gamma(1-\alpha)(1-\gamma)}}$

1, 5, 7, 35 $\quad \frac{\phi(x^5)\phi(x^7)}{\phi(x)\phi(x^{35})} = \frac{\sqrt[4]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[4]{\alpha\beta\gamma(1-\alpha)(1-\gamma)}} - \frac{\sqrt[4]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[4]{\alpha\beta\gamma(1-\gamma)(1-\delta)}}$

$$= \frac{\sqrt[4]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[4]{\alpha\beta\gamma(1-\gamma)(1-\delta)}} - \frac{\sqrt[4]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[4]{\alpha\beta\gamma(1-\alpha)(1-\gamma)}}$$

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~~$\{1 - \sqrt[3]{\alpha\beta} - \sqrt[3]{(1-\alpha)(1-\beta)}\}^3$~~.

$1 - \sqrt[3]{\alpha\beta} - \sqrt[3]{(1-\alpha)(1-\beta)} = P$ $\quad P + P_1 =$

$9\sqrt[3]{\alpha\beta(1-\alpha)(1-\beta)} = R$ $\quad P_1 = \sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)}$

~~$8.\ P^3 + 36PR = \frac{162R^2}{P} = 81R$~~

~~$8.\ P^3 + 4PR = \frac{2R^2}{P} = 9R.$~~

$P^3 - (5P + 9P_1)R - \frac{2R^2}{P} = 0$

$\frac{1}{2n} + \frac{n^2}{1(1^2+n^2)} + \frac{n^2}{2(2^2+n^2)} + \frac{n^2}{3(3^2+n^2)} + \&c$

$+ \frac{4n^2}{1^4-n^4} \cdot \frac{1}{e^{2\pi}-1} + \frac{8n^2}{2^4-n^4} \cdot \frac{1}{e^{4\pi}-1} + \frac{12n^2}{3^4-n^4} \cdot \frac{1}{e^{6\pi}-1} + \&c$

$= \frac{1}{2\pi n^2} - \frac{\pi\cot\pi n}{e^{2\pi n}-1} + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$

$\frac{n^2}{1(1^2+n^2)} + \frac{n^2}{3(3^2+n^2)} + \frac{n^2}{5(5^2+n^2)} + \&c$

$\rightarrow \left(\frac{4n^2}{1^4-n^4} \cdot \frac{1}{e^{\pi}+1} + \frac{12n^2}{3^4-n^4} \cdot \frac{1}{e^{3\pi}+1} + \&c\right).$

$= -\frac{\pi}{2} \cdot \frac{\tan\frac{\pi n}{2}}{e^{\pi n}+1} + 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{n-1}$

$\frac{1}{2n} + 2n\left\{\frac{1}{(1^2-n^2)} \cdot \frac{1}{e^{\pi}-1} - \frac{1}{3^2-n^2} \cdot \frac{1}{e^{3\pi}-1} + \&c\right\}$

$+ 2n\left\{\frac{1}{(2^2+n^2)} \cdot \frac{1}{e^{\pi}+e^{-\pi}} + \frac{1}{4^2+n^2} \cdot \frac{1}{e^{2\pi}+e^{-2\pi}} + \&c\right\}$

$= \frac{\pi}{2} \cdot \frac{\sec\frac{\pi n}{2}}{e^{\pi n}-1} + \frac{1}{n+1} - \frac{1}{n+3} + \frac{1}{n+5} - \frac{1}{n+7} + \&c$

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(1) $P = \frac{f(-x)}{x^{\frac{1}{2}} f(-x^2)}$ & $Q = \frac{f(-x^3)}{x^{\frac{3}{2}} f(-x^{15})}$ $\cdots (PQ)^2 + \left(\frac{3}{PQ}\right)^2 + 5 - 17$

$= \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3$

(2) If $P = \frac{f(-x)}{x^{\frac{1}{2}} f(-x^3)}$ & $Q = \frac{f(-x^7)}{x^{\frac{7}{2}} f(-x^{21})}$, then

$(PQ)^3 + \left(\frac{3}{PQ}\right)^3 = \left(\frac{Q}{P}\right)^4 - 7 \cdot \left(\frac{Q}{P}\right)^2 + 7 \left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^4$.

(4) If $P = \frac{f(-x^{11})}{x^{\frac{5}{2}} f(-x^{55})}$ & $Q = \frac{f(-x^3)}{x^{\frac{1}{2}} f(-x^{15})}$

$(PQ)^3 + \left(\frac{5}{PQ}\right)^3 = \left(\frac{Q}{P}\right)^6 - 9 \cdot \left(\frac{Q}{P}\right)^3 - 9 \cdot \left(\frac{P}{Q}\right)^3 - \left(\frac{P}{Q}\right)^6$.

$\frac{1}{2x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \cdots$

$= \frac{1}{x} + \frac{1}{2x^2} \cdot \frac{1}{3x} + \frac{3}{5x} + \frac{18}{7x} + \frac{60}{9x} + \cdots$

$3 = 2^2(2^2-1)/4$

$18 = 3^2(3^2-1)/4$

$60 = 4^2(4^2-1)/4$ $\cdots$

$\frac{1}{2x^3} + \frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \cdots$

$= \frac{1}{2x^2} + \frac{1}{4x^3} \cdot \frac{1}{2x} + \frac{1}{3x} + \frac{2}{x} + \frac{6}{5x} + \frac{9}{x} + \frac{18}{7x} + \cdots$

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$$\frac{x}{1+n} + \frac{1^2x^2}{3+n} + \frac{2^2x^2}{5+n} + \frac{3^2x^2}{7+n} + \&c$$

$$= 2\left(\frac{y}{m+1} - \frac{y^3}{m+3} + \frac{y^5}{m+5} - \&c\right)$$

$$\& \quad \frac{x}{2+n} + \frac{1.2x^2}{4+n} + \frac{2.3x^2}{6+n} + \frac{3.4x^2}{8+n} + \&c$$

$$= y - m\left(y + \frac{1}{y}\right)\left(\frac{y^2}{m+2} - \frac{y^4}{m+4} + \frac{y^6}{m+6} - \&c\right)$$

where $y = \frac{\sqrt{1+x^2}-1}{x}$ and $m = \frac{n}{\sqrt{1+x^2}}$.

$$2^p\left\{\frac{1}{n+p} - \frac{p}{\lfloor 1}\cdot\frac{1}{n+p+2} + \frac{p(p+1)}{\lfloor 2}\cdot\frac{1}{n+p+4} - \&c\right\}$$

$$= \frac{1}{n} + \frac{1.p}{n} + \frac{2(p+1)}{n} + \frac{3(p+2)}{n} + \&c$$

$$\frac{x}{p+n} + \frac{1.px^2}{p+2+n} + \frac{2(p+1)x^2}{p+4+n} + \frac{3(p+2)x^2}{p+6+n} + \&c$$

$$= \left(1+\frac{1}{x^2}\right)^{\frac{p-1}{2}}(2y)^p\left\{\frac{1}{m+p} - \frac{p}{\lfloor 1}\cdot\frac{y^2}{m+p+2} + \right.$$

$$\left. \frac{p(p+1)}{\lfloor 2}\cdot\frac{y^4}{m+p+4} - \&c\right\}$$

where $y = \frac{\sqrt{1+x^2}-1}{x}$ and $m = \frac{n}{\sqrt{1+x^2}}$.

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$$\frac{\pi}{2}\int_0^\infty \frac{dx}{e^{x^n}+e^{-x^n}}$$

$$= \sqrt[n]{\frac{\pi}{2}}\,\frac{\Gamma\left(\frac{1}{n}\right)}{\left\lfloor\frac{1}{n}-1\right.}\cos\frac{\pi}{2n}\int_0^\infty \frac{x^{n-2}\,dx}{e^{x^n}+e^{-x^n}}$$

$$\frac{x^2}{4n+2}+\frac{x^2}{4n+6}+\frac{x^2}{4n+10}+\&c$$

$$+\frac{2n}{x}+\frac{n-1}{1-\frac{n+1}{x}+\frac{n-2}{1-\frac{n+2}{x}-\&c}}$$

$$= 1 \text{ nearly}$$

$$\frac{1}{x}+\frac{a}{x}+\frac{a^2}{x}+\frac{a^3}{x}+\&c$$

$$= 1-\frac{ax}{1+a^2}+\frac{a^2x}{1+a^4}-\frac{a^3x}{1+a^6}+\frac{a^4x}{1+a^8}-\&c$$

nearly

$$\frac{1}{1-\frac{a}{1+a}-\frac{a^3}{1+a^2}-\frac{a^5}{1+a^4}-\&c}$$

$$= \frac{1-a^2}{1-a}\cdot\frac{1-a^5}{1-a^4}\cdot\frac{1-a^8}{1-a^7}\cdot\&c$$

$$\frac{1-a^3}{1-a}\cdot\frac{1-a^7}{1-a^5}\cdot\frac{1-a^{11}}{1-a^9}\&c = \frac{1}{1-\frac{a}{1+a^2}-\frac{a^3}{1+a^4}-\frac{a^5}{1+a^6}-\&c}$$

$$\frac{1}{1+}\,\frac{x}{1+}\,\frac{x^2+x^2}{1+}\,\frac{x^3}{1+}\,\frac{x^4+x^2}{1+}\,\frac{x^5}{1+}$$

$$= \frac{\psi(x^2)}{\psi(x)}$$

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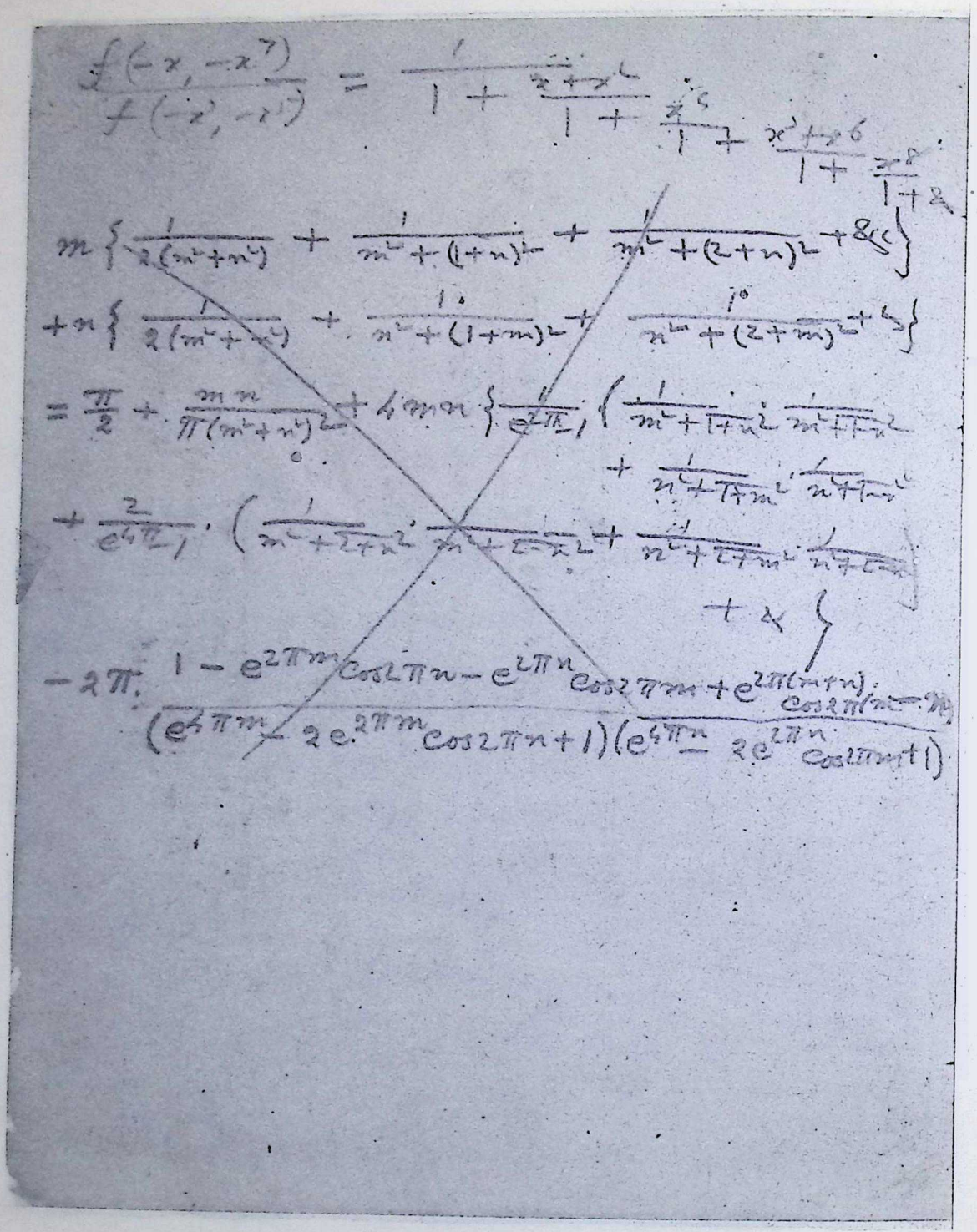

$$\frac{f(-x,-x^7)}{f(-x,-x^3)} = \cfrac{1}{1+\cfrac{x^2+x^6}{1+\cfrac{x^4}{1+\cfrac{x^6+x^{10}}{1+\cfrac{x^8}{1+\&c}}}}}$$

$$m\left\{\frac{1}{2(m^2+n^2)} + \frac{1}{m^2+(1+n)^2} + \frac{1}{m^2+(2+n)^2} + \&c\right\}$$

$$+n\left\{\frac{1}{2(m^2+n^2)} + \frac{1}{n^2+(1+m)^2} + \frac{1}{n^2+(2+m)^2} + \&c\right\}$$

$$= \frac{\pi}{2} + \frac{m+n}{\pi(m^2+n^2)} + 4mn\left\{\frac{1}{e^{2\pi}-1}\left(\frac{1}{m^2+1+n^2}\cdot\frac{1}{m^2+1+n^2}\right.\right.$$

$$\left. + \frac{1}{n^2+1+m^2}\cdot\frac{1}{n^2+1+m^2}\right)$$

$$+\frac{2}{e^{4\pi}-1}\left(\frac{1}{m^2+2+n^2}\cdot\frac{1}{m^2+2+n^2} + \frac{1}{n^2+2+m^2}\cdot\frac{1}{n^2+2+m^2}\right)$$

$$\left. + \&c\right\}$$

$$-2\pi\cdot\frac{1-e^{2\pi m}\cos 2\pi n - e^{2\pi n}\cos 2\pi m + e^{2\pi(m+n)}\cos 2\pi(m-n)}{(e^{4\pi m}-2e^{2\pi m}\cos 2\pi n+1)(e^{4\pi n}-2e^{2\pi n}\cos 2\pi m+1)}$$

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If $\phi(\alpha,\beta) = \alpha\left\{\frac{1}{2(\alpha^2+\beta^2)} + \frac{1}{\alpha^2+(1+\beta)^2} + \frac{1}{\alpha^2+(2+\beta)^2} + \&c\right\}$

$- 4\alpha\beta\left\{\frac{1}{e^{2\pi}-1}\cdot\frac{1}{\alpha^2+1+\beta^2}\cdot\frac{1}{\alpha^2+1-\beta^2} + \frac{2}{e^{4\pi}-1}\cdot\frac{1}{\alpha^2+2+\beta^2}\cdot\frac{1}{\alpha^2+2-\beta^2}\right\}$

$+ \frac{\pi}{e^{4\pi\alpha} - 2e^{2\pi\alpha}\cos 2\pi\beta + 1}$.

then $\phi(\alpha,\beta) + \phi(\beta,\alpha) = \frac{\pi}{2} + \frac{\alpha\beta}{\pi(\alpha^2+\beta^2)^2}$

$+ \frac{\pi}{2}\cdot\frac{\cosh 2\pi(\alpha-\beta) - \cos 2\pi(\alpha-\beta)}{(\cosh 2\pi\alpha - \cos 2\pi\beta)(\cosh 2\pi\beta - \cos 2\pi\alpha)}$.

If $\phi(\alpha,\beta) = \alpha\left\{\frac{1}{\alpha^2+(1+\beta)^2} + \frac{1}{\alpha^2+(3+\beta)^2} + \frac{1}{\alpha^2+(5+\beta)^2} + \&c\right\}$

$+ 4\alpha\beta\left\{\frac{1}{e^{\pi}+1}\cdot\frac{1}{\alpha^2+1+\beta^2}\cdot\frac{1}{\alpha^2+1-\beta^2} + \&c\right\}$

$+ \frac{\pi/2}{e^{2\pi\alpha} + 2e^{\pi\alpha}\cos\pi\beta + 1}$, then

$\phi(\alpha,\beta) + \phi(\beta,\alpha) = \frac{\pi}{4} + \frac{\pi}{4}\cdot\frac{\cosh\pi(\alpha-\beta) - \cos\pi(\alpha-\beta)}{(\cosh\pi\alpha + \cos\pi\beta)(\cosh\pi\beta + \cos\pi\alpha)}$

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$$\frac{1}{4} + \frac{1}{1+(2n)^2}\cdot\frac{1}{e^{\pi m}+e^{-\pi m}} + \frac{1}{1+(4n)^2}\cdot\frac{1}{e^{2\pi m}+e^{-2\pi m}}$$

$$+m\left\{\frac{1}{n^2-m^2}\cdot\frac{1}{e^{\frac{\pi}{m}}-1} - \frac{1}{(3n)^2-m^2}\cdot\frac{1}{e^{\frac{2\pi}{m}}-1} + \&c\right\}$$

$$= \frac{\pi}{4n}\cdot\frac{\sec\frac{\pi m}{2n}}{e^{\frac{\pi}{n}}-1} + \frac{1}{2}\left(\frac{1}{n+m} - \frac{1}{3n+m} + \frac{1}{5n+m} - \&c\right)$$

$$\frac{x}{8\pi} + \frac{\sin x}{1(e^{2\pi}-1)} + \frac{\sin 4x}{2(e^{4\pi}-1)} + \frac{\sin 9x}{3(e^{6\pi}-1)} +$$

$$= \frac{1}{4}\left\{\frac{B_2}{1\lfloor 1}x - \frac{B_6}{3\lfloor 3}x^3 + \frac{B_{10}}{5\lfloor 5}x^5 - \&c\right\}$$

If $\alpha\beta = \frac{\pi^2}{2}$

$$\alpha^2\left\{\frac{\operatorname{sech}\frac{\pi}{2}}{\cosh\alpha+\cos\alpha} - \frac{3^3\operatorname{sech}\frac{3\pi}{2}}{\cosh 3\alpha+\cos 3\alpha} + \&c\right\}$$

$$= \beta^2\left\{\frac{\operatorname{sech}\frac{\pi}{2}}{\cosh\beta+\cos\beta} - \frac{3^3\operatorname{sech}\frac{3\pi}{2}}{\cosh 3\beta+\cos 3\beta} + \&c\right.$$

$$\int_0^\infty \frac{\sin 2nx}{x(\cosh\pi x+\cos\pi x)}$$

$$= \frac{\pi}{4} - 2\left\{\frac{e^{-n}\cos n}{\cosh\frac{\pi}{2}} - \frac{e^{-3n}\cos 3n}{3\cosh\frac{3\pi}{2}} + \&c\right\}$$

If $\alpha\beta = \frac{\pi^2}{4}$, then

$$\frac{1}{\cosh\alpha+\cos\alpha} - \frac{1}{3(\cosh 3\alpha+\cos 3\alpha)} + \&c$$

$$+\frac{2\cos\beta\cosh\beta}{\cosh\frac{\pi}{2}(\cosh 2\beta+\cos 2\beta)} - \frac{2\cos 3\beta\cosh 3\beta}{3\cosh\frac{3\pi}{2}(\cosh 6\beta+\cos 6\beta)}$$

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$$+\frac{2\cos 5A\cosh 5A}{5^5\cosh^5\frac{\pi}{2}(\cosh 10A+\cos 10A)}-\&c=\frac{\pi}{8}.$$ 180

If $\alpha\beta=\frac{\pi^2}{2}$, then

$$\frac{\cos\alpha}{\cosh\alpha-\cos\alpha}-\frac{\cos 3\alpha}{3^3(\cosh 3\alpha-\cos 3\alpha)}+\&c$$

$$=\frac{\pi^3}{32\alpha^2}-\frac{\pi}{8}+\frac{\sin\beta\sinh\beta}{\cosh 2\beta+\cos 2\beta}\cdot\frac{\coth\pi}{1}$$

$$+\frac{\sin 2\beta\sinh 2\beta}{\cosh 4\beta+\cos 4\beta}\cdot\frac{\coth 2\pi}{2}+\&c$$

$$\frac{B_2}{1.2.2n}+\frac{B_4}{3.4.2^2n^3}-\frac{B_6}{5.6.2^3n^5}\cdots-\frac{B_{2r}}{7.8.2^3n^7}+\&c$$

$$=\cancel{n+\frac{\pi}{4}n+\log\Gamma n}$$

$$=\log\frac{e^n\Gamma n}{n^n\sqrt{2\pi n}}+n\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)$$

$$-\frac{1}{2}\log\left[\sqrt{2}\left\{1+\left(\frac{n}{n+1}\right)^2\right\}\left\{1+\left(\frac{n}{n+2}\right)^2\right\}\&c\right].$$

$$\left\{1+\left(\frac{n}{1}\right)^3\right\}\left\{1+\left(\frac{n}{2}\right)^3\right\}\left\{1+\left(\frac{n}{3}\right)^3\right\}\&c$$

$$\times\left\{1+3\cdot\left(\frac{n}{n+2}\right)^2\right\}\left\{1+3\cdot\left(\frac{n}{n+4}\right)^2\right\}\left\{1+3\cdot\left(\frac{n}{n+6}\right)^2\right\}\&c$$

$$=\frac{\Gamma\frac{n}{2}-1}{\Gamma\frac{n-1}{2}}\cdot\frac{\cosh\pi n\sqrt{3}-\cos\pi n}{2^{n+2}\cdot\pi n\sqrt{\pi}}$$

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$\frac{1}{2}\log(2\pi x) + \frac{1}{3}\log\left(1+\frac{x^2}{1^2}\right)\left(1+\frac{x^2}{2^2}\right)\left(1+\frac{x^2}{3^2}\right)\cdots$

$= \frac{2\pi x}{3\sqrt{3}} + \frac{B_4}{3.4\,x^3} - \frac{B_{10}}{9.10\,x^9} + \frac{B_{16}}{15.16.x^{15}} - \cdots$

$\frac{1}{2}\log(2\pi n) + \frac{1}{3}\log\left(1+\frac{n^2}{1^2}\right)\left(1+\frac{n^2}{2^2}\right)\left(1+\frac{n^2}{3^2}\right)\cdots$

$= \frac{1}{3}\log\left(e^{\pi n\sqrt{3}} - 2\cos\pi n + e^{-\pi n\sqrt{3}}\right)$

$- \frac{\pi n}{3\sqrt{3}} + \frac{B_4}{3.4\,n^3} - \frac{B_{10}}{9.10\,n^9} + \cdots$

$\left\{1+\left(\frac{m+n}{1+m}\right)^3\right\}\left\{1+\left(\frac{m+n}{2+m}\right)^3\right\}\left\{1+\left(\frac{m+n}{3+m}\right)^3\right\}\cdots$

$\times\left\{1+\left(\frac{m+n}{1+n}\right)^3\right\}\left\{1+\left(\frac{m+n}{2+n}\right)^3\right\}\left\{1+\left(\frac{m+n}{3+n}\right)^3\right\}\cdots$

$= \frac{(\lfloor m)^3(\lfloor n)^3}{\lfloor 2m+n \;\; \lfloor 2n+n} \cdot \frac{\cosh\pi(m+n)\sqrt{3} - \cos\pi(m-n)}{2\pi^2(m^2+mn+n^2)}$

$\int_0^\infty \log(1+x^2)\cos nx\,dx = -\frac{\pi}{n}e^{-n}$

$\log\left(1+\frac{x^2}{1^2}\right) - 3\log\left(1+\frac{x^2}{2^2}\right) + 5\log\left(1+\frac{x^2}{5^2}\right) - \cdots$

$= \frac{4}{\pi}\cdot\left\{\frac{1-e^{-\frac{\pi x}{2}}}{1^2} - \frac{1-e^{-3\frac{\pi x}{2}}}{3^2} + \frac{1-e^{-5\frac{\pi x}{2}}}{5^2} - \cdots\right\}$

$- 2x\tan^{-1}e^{-\frac{\pi x}{2}}.$

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$$\log\left(1-\frac{x^2}{1^2}\right)-3\log\left(1-\frac{x^2}{3^2}\right)+5\log\left(1-\frac{x^2}{5^2}\right)-\&c \qquad 181$$

$$=\frac{4}{\pi}\left\{\frac{1-\cos\frac{\pi x}{2}}{1^2}-\frac{1-\cos\frac{3\pi x}{2}}{3^2}+\&c\right\}$$

$$+2C\log\tan\frac{\pi-\pi x}{4}$$

$$=\frac{4}{\pi}\left\{\frac{1-\tan\left(\frac{\pi}{4}-\frac{\pi x}{4}\right)}{1^2}-\frac{1-\tan^3\left(\frac{\pi}{4}-\frac{\pi x}{4}\right)}{3^2}+\&c\right\}$$

$$+\log\tan\frac{\pi-\pi x}{4}.$$

If $\frac{\pi\alpha}{2}=\log\tan\left(\frac{\pi}{4}+\frac{\pi\beta}{4}\right)$

then $\log\left(1+\frac{\alpha^2}{1^2}\right)-3\log\left(1+\frac{\alpha^2}{3^2}\right)+5\log\left(1+\frac{\alpha^2}{5^2}\right)-\&c$

$$=\frac{\pi\alpha\beta}{2}+\log\left(1-\frac{\beta^2}{1^2}\right)-3\log\left(1-\frac{\beta^2}{3^2}\right)+5\log\left(1-\frac{\beta^2}{5^2}\right)-\&c$$

$$\int_0^\infty \frac{\sinh ax}{\sinh\pi x}\cdot\frac{dx}{a^2+x^2}=\int_0^1 \frac{x^{n-1}}{n}\cdot\frac{\sin a\,dx}{1+2x\cos a+x^2}$$

$$\int_0^\infty \frac{\sin ax}{\sinh\frac{\pi x}{2}}\cdot\frac{dx}{1+x^2}=\frac{\pi}{2}\sinh a$$

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$$\frac{\phi^2(e^{-\pi\sqrt{\frac{n}{3}}})}{\phi^2(e^{-\pi\sqrt{3n}})} = \sqrt{3}\, a_n.$$

$a_1 = 1$; $a_5 = \frac{\sqrt{5}-1}{2}$; $a_7 = \frac{\sqrt{3}+1}{\sqrt{2}}\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{\frac{3}{2}}$.

$$\frac{\phi^4(e^{-\pi\sqrt{\frac{n}{3}}})}{\phi^4(e^{-\pi\sqrt{3n}})} = 3a_n.$$

$a_1 = 1$; $a_5 = \left(\frac{\sqrt{5}-1}{2}\right)^2$; $a_7 = (2+\sqrt{3})\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3$;

$a_{11} = (10-3\sqrt{11})(2\sqrt{3}+\sqrt{11})$; $a_{13} =$

$$n\, e^{-\frac{\pi}{4}(n-1)\sqrt{\frac{m}{n}}}\, \frac{\psi^2(e^{-\pi\sqrt{mn}})\,\phi^2(e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{\frac{m}{n}}})\,\phi^2(e^{-2\pi\sqrt{\frac{m}{n}}})}$$

$= a_{m,n}$.

$a_{3,5} = \frac{3-\sqrt{5}}{2}$; $a_{3,7} = 2-\sqrt{3}$; $a_{3,11} = 2\sqrt{3}-\sqrt{11}$,

$a_{3,13} = \left(\sqrt{\frac{5+\sqrt{13}}{8}} - \sqrt{\frac{\sqrt{13}-3}{8}}\right)^8$; $a_{3,19} = 2\sqrt{19}-5\sqrt{3}$.

$a_{3,23} = \left(\sqrt{\frac{7+\sqrt{3}}{2}} - \sqrt{\frac{5+\sqrt{3}}{2}}\right)^2$; $a_{3,31} = (2-\sqrt{3})^3$.

$a_{3,59} = 102\sqrt{3} - 23\sqrt{59}$; $a_{3,71} = \left(\sqrt{\frac{175+100\sqrt{3}}{2}} - \sqrt{\frac{173+100\sqrt{3}}{2}}\right)^8$

$a_{5,9} = (2-\sqrt{3})^2$; $a_{5,11} = \left(\sqrt{\frac{7+\sqrt{5}}{8}} - \sqrt{\frac{\sqrt{5}-1}{8}}\right)$;

$a_{5,13} = \left(\sqrt{\frac{9+\sqrt{65}}{2}} - \sqrt{\frac{7+\sqrt{65}}{2}}\right)^2$; $a_{5,17} = (\sqrt{17}-4)^2$;

$a_{5,29} = \left(\sqrt{49+4\sqrt{145}} - \sqrt{48+4\sqrt{145}}\right)^2$; &

$a_{7,9} = \left(\sqrt{\frac{5+\sqrt{21}}{8}} - \sqrt{\frac{\sqrt{21}-3}{8}}\right)^8$

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$a_{3,3} = \frac{1}{\sqrt{3}}$; $a_{3,9} = \frac{1}{(\sqrt[3]{2}+1)^2}$; $a_{3,16} = \frac{2-\sqrt{3}}{3}$

$$\frac{\phi^2(e^{-\pi})}{\phi^2(e^{-n\pi})} = n\, b_n \sqrt[8]{\alpha\beta(1-\beta)}$$

$b_1 = 1$; $b_3 = \frac{1}{\sqrt{3}}$; $b_5 = 1$; $b_7 = \frac{\sqrt{3+\sqrt{7}}}{\sqrt{2}\sqrt[8]{7}}$.

$b_{11} =$

$$\frac{x}{1+x} - \frac{2^2x^3}{1+x^2} + \frac{3^2x^6}{1+x^3} - \frac{4^2x^{10}}{1+x^4} + \&c$$

$$= \phi^2(-x)\left\{x \cdot \frac{1+x}{(1-x)^2} + x^6 \cdot \frac{1+x^2}{(1-x^3)^2} + x^{10}\frac{1+x^5}{(1-x^5)^2} + \&c\right\}$$

$$\frac{1+x}{1-x} - 3^2x^2 \cdot \frac{1+x^3}{1-x^3} + 5^2x^6 \cdot \frac{1+x^5}{1-x^5} - \&c$$

$$= \psi^2(x)\left\{1 - \frac{3x^2}{(1+x)^2} + \frac{5x^6}{(1+x^2)^2} - \frac{9x^{12}}{(1+x^3)^2} + \&c\right\}$$

$$x\,\psi(\alpha)\psi(\alpha^4) = \frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^6}{1-x^5} - \frac{x^{10}}{1-x^7} + \&c$$

$$\frac{x^n}{\lfloor n} \left\{1 + \frac{x^2}{12}\cdot\frac{1}{3+n} + \frac{x^4}{12}\cdot\frac{1}{(1+n)(2+n)} + \&c\right\}$$

$$- \frac{x^{-n}}{\lfloor -n} \left\{1 + \frac{x^2}{12}\cdot\frac{1}{1-n} + \frac{x^4}{12}\cdot\frac{1}{(1-n)(2-n)} + \&c\right\}$$

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If n is a positive integer

$$\frac{1^{4n}}{(e^{\pi}-e^{-\pi})^2}+\frac{2^{4n}}{(e^{2\pi}-e^{-2\pi})^2}+\&c=\frac{n}{\pi}\left(\frac{B_{4n}}{8n}+\frac{1^{4n-1}}{e^{2\pi}-1}+\frac{2^{4n-1}}{e^{4\pi}-1}+\right)$$

$$\frac{1}{2x^2}+\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\&c$$

$$=\frac{1}{2\pi x^3}+\frac{\pi}{3x}-\frac{\pi^2}{\sin^2\pi x\,(e^{2\pi x}-1)}$$

$$+4x\left\{\frac{1}{e^{2\pi}-1}\cdot\frac{1}{(1^2-x^2)^2}+\frac{2}{e^{4\pi}-1}\cdot\frac{1}{(2^2-x^2)^2}+\&c\right\}$$

$$+8\pi x^3\left\{\frac{1}{(e^{\pi}-e^{-\pi})^2}\cdot\frac{1}{1^4-x^4}+\frac{1}{(e^{2\pi}-e^{-2\pi})^2}\cdot\frac{1}{2^4-x^4}+\&c\right\}$$

If $\theta=\frac{u}{\sqrt{2}}=v+\frac{1}{2}\cdot\frac{v^5}{5}+\frac{1.3}{2.4}\cdot\frac{v^9}{9}+\&c$, then

$$\frac{u^2}{2\theta^2}=\frac{1}{\sin^2\theta}-\frac{1}{\pi}-8\left(\frac{\cos 2\theta}{e^{2\pi}-1}+\frac{2\cos 4\theta}{e^{4\pi}-1}+\&c\right).$$

$$\cot\theta+\frac{\theta}{\pi}+4\left(\frac{\sin 2\theta}{e^{2\pi}-1}+\frac{\sin 4\theta}{e^{4\pi}-1}+\&c\right)$$

$$=\frac{u}{\sqrt{2}}\left\{\frac{1}{v}-\frac{1}{2}\cdot\frac{v^3}{3}-\frac{1.3}{2.4}\cdot\frac{v^7}{7}-\&c\right\}$$

$$\log\sin\theta+\frac{\theta^2}{2\pi}+2\left\{\frac{\cos 2\theta}{1(e^{2\pi}-1)}+\frac{\cos 4\theta}{2(e^{4\pi}-1)}+\&c\right\}$$

$$=\log\frac{v\sqrt{2}}{u}+\frac{1}{3}\cdot\frac{v^4}{4}+\frac{1.5}{3.7}\cdot\frac{v^8}{8}+\frac{1.5.9}{3.7.11}\cdot\frac{v^{12}}{12}+\&c$$

$$\frac{1}{2}\tan^{-1}v=\frac{\sin\theta}{\cosh\frac{\pi}{2}}+\frac{\sin 3\theta}{3\cosh\frac{3\pi}{2}}+\frac{\sin 5\theta}{5\cosh\frac{5\pi}{2}}+\&c.$$

$$\frac{1}{2}\cos^{-1}v^2=\frac{\cos\theta}{\cosh\frac{\pi}{2}}-\frac{\cos 3\theta}{3\cosh\frac{3\pi}{2}}+\frac{\cos 5\theta}{5\cosh\frac{5\pi}{2}}-\&c$$

$$\frac{\pi\theta}{8}-\frac{\sin\theta}{1^2\cosh\frac{\pi}{2}}+\frac{\sin 3\theta}{3^2\cosh\frac{3\pi}{2}}-\&c$$

$$=\frac{\sqrt{2}}{4\mu}\left\{\frac{v^3}{3}+\frac{2}{8}\cdot\frac{v^7}{7}+\frac{2.4}{3.5}\cdot\frac{v^{11}}{11}+\&c\right\}.$$

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If $\theta \frac{\pi}{2} = v - \frac{1}{2}\cdot\frac{v^5}{5} + \frac{1.3}{2.4}\cdot\frac{v^9}{9} - \&c$

$2\tan^{-1} v = \theta + \frac{\sin 2\theta}{\cosh \pi} + \frac{\sin 4\theta}{2\cosh 2\pi} + \frac{\sin 6\theta}{3\cosh 3\pi} + \&c$

$\frac{\pi}{8} - \frac{1}{2}\tan^{-1} v^2 = \frac{\cos\theta}{\cosh \frac{\pi}{2}} - \frac{\cos 3\theta}{3\cosh \frac{3\pi}{2}} + \frac{\cos 5\theta}{5\cosh \frac{5\pi}{2}} - \&c$

$\frac{1}{2}\log\frac{1+v}{1-v} = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + 4\left\{\frac{\sin\theta}{e^{\pi}-1} - \frac{\sin 3\theta}{3(e^{3\pi}-1)} + \&c\right\}$

$\frac{1}{2\cdot\pi\cdot x^4} + \coth\pi\left(\frac{1}{1+x^2+x^4} + \frac{1}{1-x^2+x^4}\right)$

$+ 2\coth 2\pi\left(\frac{1}{16+4x^2+x^4} + \frac{1}{16-4x^2+x^4}\right)$

$+ \&c \qquad \&c$

$= \frac{\pi}{x^2\sqrt{3}}\cdot\frac{\sinh \pi x\sqrt{3}\sinh \pi x + \sin \pi x\sqrt{3}\sin \pi x}{(\cosh \pi x\sqrt{3} - \cos \pi x)(\cosh \pi x - \cos \pi x\sqrt{3})}$

$\Sigma\frac{1}{x} - \Sigma\frac{1}{x+3} + \frac{1}{x} - \log 3$

$= \frac{2}{3}\cdot\frac{1}{x^2} + \frac{2^3-2}{6} + \frac{4^3-4}{3x^2} + \frac{5^3-5}{6} + \frac{7^3-7}{5x^2} + \&c$

$\frac{16n}{\left(e^{\frac{\pi\sqrt{3}}{2}} + e^{-\frac{\pi\sqrt{3}}{2}}\right)^2} - \frac{2^{6n}}{\left(e^{\pi\sqrt{3}} - e^{-\pi\sqrt{3}}\right)^2} + \&c$

$+ \frac{n\sqrt{3}}{\pi}\left\{\frac{B_{6n}}{12n}\cos 3\pi n - \left(\frac{16^{n-1}}{e^{\pi\sqrt{3}}+1} - \&c\right)\right\} = 0$

n being a positive integer

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$$2^2\left\{1+240\left(\frac{1^3}{e^{2\pi\sqrt{2}}-1}+\frac{2^3}{e^{4\pi\sqrt{2}}-1}+\&c\right)\right\}-\left\{1+240\left(\frac{1^3}{e^{\pi\sqrt{2}}-1}+\frac{2^3}{e^{2\pi\sqrt{2}}}\right.\right.$$

$$=0$$

$$2^3\left\{1-504\left(\frac{1^5}{e^{\pi\sqrt{2}}-1}+\&c\right)\right\}+\left\{1-504\left(\frac{1^5}{e^{\pi\sqrt{2}}-1}+\&c\right)\right\}=0$$

$$2^5\left\{1+480\left(\frac{1^7}{e^{\pi\sqrt{2}}-1}+\&c\right)\right\}-\left\{1+480\left(\frac{1^7}{e^{\pi\sqrt{2}}-1}+\&c\right)\right\}=0$$

If $S_n=\frac{B_n}{2n}+\frac{1^{n-1}}{e^{2\pi}-1}+\frac{2^{n-1}}{e^{4\pi}-1}+\&c$

where n is a multiple of 4;

then $\frac{(n+3)(n-4)}{1\cdot 24}\cdot S_{n+2}=\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}S_4S_{n-2}$

$+\frac{n(n-1)\ldots(n-7)}{\lfloor 8}S_8S_{n-6}+\&c$

If $\frac{2}{3}\theta u=v+\frac{1}{2}\cdot\frac{v^7}{7}+\frac{1.3}{2.4}\cdot\frac{v^{13}}{13}+\&c$

then $\frac{4}{9}\frac{u^2}{v^2}=\frac{1}{\sin^2\theta}-\frac{2}{\pi\sqrt{3}}+8\left(\frac{\cos 2\theta}{e^{\pi\sqrt{3}}+1}-\frac{2\cos 4\theta}{e^{2\pi\sqrt{3}}-1}\right.$

$\left.+\&c\right)$

where $k=\frac{\sqrt{\pi}}{\lfloor -\frac{1}{6}\rfloor -\frac{1}{3}}$.

$$\frac{1}{2}\log\left[\left\{1+\left(\frac{x}{n+1}\right)^2\right\}\left\{1+\left(\frac{x}{n+2}\right)^2\right\}\left\{1+\left(\frac{x}{n+3}\right)^2\right\}\&c\right]$$

$$=\log\lfloor n+n+x\tan^{-1}\frac{x}{n}-\frac{n}{2}\log(n^2+x^2)$$

$$-\frac{1}{2}\log(2\pi\sqrt{n^2+x^2})-\int_0^\infty\frac{\tan^{-1}\frac{2nz}{n^2+x^2-z^2}}{e^{2\pi z}-1}dz.$$

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$$\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots\ldots+\sqrt{x}$$

$$= c+\frac{2}{3}x\sqrt{x}+\frac{1}{2}\sqrt{x}+\frac{1}{6}\left\{\frac{1}{(\sqrt{x}+\sqrt{x+1})^3}+\frac{1}{(\sqrt{x+1}+\sqrt{x+2})^3}+\&c\right\}$$

$$1\sqrt{1}+2\sqrt{2}+3\sqrt{3}+\ldots\ldots+x\sqrt{x}$$

$$= c+\frac{2}{5}x^2\sqrt{x}+\frac{x}{2}\sqrt{x}+\frac{\sqrt{x}}{8}+\frac{1}{40}\left\{\frac{1}{(\sqrt{x}+\sqrt{x+1})^5}+\frac{1}{(\sqrt{x+1}+\sqrt{x+2})^5}+\&c\right\}$$

$$(1^2\sqrt{1}+2^2\sqrt{2}+3^2\sqrt{3}\quad\ldots\quad+x^2\sqrt{x})+\frac{1}{16}(\sqrt{1}+\sqrt{2}+\ldots\ldots+\sqrt{x})$$

$$= c+\frac{2}{7}x^3\sqrt{x}+\frac{x^2}{3}\sqrt{x}+\frac{x\sqrt{x}}{4}+\frac{\sqrt{x}}{32}+\frac{1}{224}\left\{\frac{1}{(\sqrt{x}+\sqrt{x+1})^7}+\&c\right\}$$

$$1^3\sqrt{1}+2^3\sqrt{2}\ldots\ldots+x^3\sqrt{x}+$$

$$c+\frac{\pi}{3}\log x+\frac{1}{2x}-\frac{1}{4\pi x^2}+2\left(\frac{1}{e^{2\pi}-1}\cdot\frac{1}{1^2+x^2}+\frac{2}{e^{4\pi}-1}\cdot\frac{2}{2^2+x^2}+\&c\right)$$

$$+\frac{\pi\cot\pi x}{e^{2\pi x}-1}+\frac{2\pi\log(2\sin\pi x)}{(e^{\pi x}-e^{-\pi x})^2}-2\pi\left\{\frac{\log(1^4+x^4)}{(e^{\pi}-e^{-\pi})^2}+\frac{\log(2^4+x^4)}{(e^{2\pi}-e^{-2\pi})^2}+2\&c\right\}$$

$$-2\pi\sum_{n=1}^{n=\infty}e^{-2\pi nx}\left[-\left\{\frac{2\pi x}{1+x^2}+\frac{2\sin4\pi x}{2^2+x^2}+\&c\right\}\right.$$

$$\left.-x^3\left\{\frac{\cos2\pi x}{1(1^2+x^4)}+\frac{\cos4\pi x}{2(2^2+x^4)}+\&c\right\}\right]$$

$$= 1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{x}.$$

$$y\sqrt{\frac{\sqrt{13}-3}{2}}\quad\text{where}\quad\sqrt{5}\left(y^3+y^2\frac{\sqrt{13}-1}{2}+y\cdot\frac{\sqrt{13}+1}{2}-1\right)$$

$$\pm\left\{y^3+y^2\left(\frac{\sqrt{13}+1}{2}\right)^2+y\left(\frac{\sqrt{13}-1}{2}\right)^2+1\right\}=0$$

$$\sqrt{765}\quad(\sqrt{5}-2)^8\left(\frac{\sqrt{85}-9}{2}\right)^6\cdot(4-\sqrt{15})^6(16-\sqrt{265})^2$$

$$\times\left(\sqrt{\frac{22+3\sqrt{51}}{4}}-\sqrt{\frac{18+3\sqrt{51}}{4}}\right)^{12}\left(\sqrt{\frac{10+\sqrt{51}}{4}}-\sqrt{\frac{6+\sqrt{51}}{4}}\right)^2$$

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If $\phi = \frac{x}{1+\frac{x^5}{1+\frac{x^{10}}{1+\&c}}}$ $\quad f = \frac{x}{1+\frac{x}{1+\frac{x^2}{1+\frac{x^2}{1+\&c}}}}$

then $f^5 = \phi \cdot \frac{1-2\phi+4\phi^2-3\phi^3+\phi^4}{1+3\phi+4\phi^2+2\phi^3+\phi^4}$

$\sqrt{505}\,(\sqrt{5}-2)^{14}(\sqrt{101}-10)^6 \left(\frac{5\sqrt{5}+\sqrt{101}}{4} \pm \sqrt{\frac{105+5\sqrt{505}}{8}}\right)^{12}$

$\sqrt{38}\quad \frac{g^3+g\sqrt{2}}{1+g^2\sqrt{2}} = \sqrt{1+\sqrt{2}}$

$\sqrt{26}\quad \frac{g^3+g\cdot\frac{\sqrt{13}+1}{2}}{1+g^2\cdot\frac{\sqrt{13}-1}{2}} = \sqrt{\frac{3+\sqrt{13}}{2}}$

$\sqrt{50}\quad \frac{g^3-g^2}{1+g} = \frac{\sqrt{5}+1}{2}$.

$\frac{a}{x+\frac{a^2}{x+\frac{a^3}{x+\frac{a^4}{x+\&c}}}} = 1 - \frac{ax}{1+\frac{a^2}{1-\frac{a^2x}{1+\frac{a^4}{1-\&c}}}}$ nearly.

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$\left(\frac{1}{64}\right)$. $\sqrt{23}$, $1-g^3=g^2$

$\sqrt{31}$ $1-g^3=g$

(64.) $\sqrt{11}$ $\frac{1}{2}-g^3=g-g^2$

$\sqrt{19}$ $\frac{1}{2}-g^3=g^2$

$\sqrt{27}$ $\frac{1}{2}-g^3=g^2\sqrt{3}$.

$\sqrt{43}$ $\frac{1}{2}-g^3=g$ $\sqrt{67}$ $\frac{1}{2}-g^3=g^2+g$.

$\sqrt{3}$ $2-\sqrt{3}$

$\sqrt{5}$ $\left(\sqrt{\frac{3+\sqrt{5}}{4}}-\sqrt{\frac{\sqrt{5}-1}{4}}\right)^4$

$\sqrt{7}$ $8-3\sqrt{7}$

$\sqrt{9}$ $\left(\sqrt{\frac{3+\sqrt{3}}{2}}-\sqrt{\frac{\sqrt{3}-1}{2}}\right)^8$

$\sqrt{13}$ $\left(\sqrt{\frac{7+\sqrt{13}}{4}}-\sqrt{\frac{2+\sqrt{13}}{4}}\right)^4$

$\sqrt{15}$ $(2-\sqrt{3})^2(4-\sqrt{15})$

$\sqrt{17}$ $\left(\sqrt{\frac{3+\sqrt{4+\sqrt{17}}}{4}}-\sqrt{\frac{\sqrt{4+\sqrt{17}}-1}{4}}\right)^8$

$\sqrt{21}$

$\sqrt{25}$ $\left(\sqrt{\frac{5+\sqrt{5}}{4}}-\sqrt{\frac{\sqrt{5}+1}{4}}\right)^8$

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213 168 156

$\sqrt{7}$ ——— $8-2\sqrt{7}$

$\sqrt{15}$ ——— $(2-\sqrt{3})^2(4-\sqrt{15})$

$\sqrt{39}$ ———

$\sqrt{55}$ ——— $(10-3\sqrt{11})(3\sqrt{5}-2\sqrt{11})\left(\sqrt{\frac{5+\sqrt{5}}{2}}-\sqrt{\frac{5-\sqrt{5}}{2}}\right)$

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$$x \cot x = 1 - B_2 \frac{2x^2}{\lfloor 2} - B_4 \frac{x^4}{\lfloor 4} - B_6 \frac{2^6}{\lfloor 6} - \&c$$

$$1 - x\left\{\cot \frac{x}{2} + \cot \frac{x\omega}{2} + \cot \frac{x\omega^2}{2}\right\}$$

$$= 6\left\{\frac{B_4}{\lfloor 4} x^4 + \frac{B_{10}}{\lfloor 10} x^{10} + \frac{B_{16}}{\lfloor 16} x^{16} + \&c\right\}$$

$$= x \cdot \frac{\frac{x^6}{\lfloor 6} - \frac{x^{12}}{\lfloor 12} + \&c}{\frac{x^3}{\lfloor 3} - \frac{x^9}{\lfloor 9} + \&c}$$

(b) $$-\frac{x}{2}\left\{\cot \frac{x}{2} + \omega^2 \cot \frac{x\omega^2}{2} + \omega \cot \frac{x\omega}{2}\right\}$$

$$= 3\left\{\frac{B_2}{\lfloor 2} x^2 + \frac{B_8}{\lfloor 8} x^8 + \&c\right\}$$

$$= x \cdot \frac{\frac{x^4}{\lfloor 4} - \frac{x^{10}}{\lfloor 10} + \&c}{\frac{x^3}{\lfloor 3} - \frac{x^9}{\lfloor 9} + \&c}$$

(c) $$-\frac{x}{2}\left\{\cot \frac{x}{2} + \omega \cot \frac{x\omega}{2} + \omega^2 \cot \frac{x\omega^2}{2}\right\}$$

$$= 3\left\{B_0 + \frac{B_6}{\lfloor 6} x^6 + \frac{B_{12}}{\lfloor 12} x^{12} + \&c\right\}$$

$$= -x \cdot \frac{\frac{x^2}{\lfloor 2} - \frac{x^8}{\lfloor 8} + \&c}{\frac{x^3}{\lfloor 3} - \frac{x^7}{\lfloor 7} + \&c}$$

$$B_n - \frac{n}{4} \cdot \frac{n-1}{5} \cdots \frac{n-5}{9} B_{n-6} + \frac{n}{4} \cdot \frac{n-1}{5} \cdots \frac{n-11}{15} B_{n-12}$$

$$+ \&c \quad \text{~~of~~} = \frac{2}{(n+1)(n+2)} (-1)^{\frac{n-2}{6}} \text{ or}$$

$$\frac{2}{(n+1)(n+4)} (-1)^{\frac{n-6}{6}} \text{ or}$$

$$\frac{4}{(n+1)(n+2)} (-1)^{\frac{n-4}{6}}$$

44B

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$B_2 = \frac{1}{6}$

$B_8 - \frac{B_2}{3} = -\frac{1}{45}$

$B_4 - \frac{143}{4} B_8 + \frac{B_2}{5} = \frac{1}{120}$

$B_4 = \frac{1}{30}$

$B_{10} - \frac{5}{2} B_4 = -\frac{1}{132}$

$B_{16} - \frac{286}{3} B_{10} + 4 B_4 = \frac{1}{306}$

$B_0 = -1$

$B_6 - \frac{B_0}{84} = \frac{1}{28}$

$B_{12} - 11 B_6 + \frac{B_0}{455} = -\frac{1}{91}$

$B_{18} - 221 B_{12} + \frac{204}{5} B_6 - \frac{B_0}{1330} = \frac{1}{190}$

$\frac{n+2}{3} B_n = {}^nC_6 B_{n-6} B_6 + {}^nC_{12} B_{n-12} B_{12} + \dots$

when $n-2$ is a multiple of 6.

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i $B_n - \frac{n(n-1)}{2.3} B_{n-2} + \frac{n(n-1)(n-2)(n-3)}{4.3.4.5} B_{n-4} - \&c. + \frac{(-1)^{\frac{n}{2}}}{2} = 0$

ii $B_n - \frac{n(n-1)}{3.4} B_{n-2} + \frac{n(n-1)(n-2)(n-3)}{3.4.5.6} B_{n-4} - \&c. + \frac{(-1)^{\frac{n}{2}}}{n+1} = 0$

iii $B_n - \frac{n(n-1)}{4.6} B_{n-2} + \frac{n(n-1)(n-2)(n-3)}{4.6.8.10} B_{n-4} - \&c. + \frac{(-1)^{\frac{n}{2}}}{2^n} = 0$

N.B. 0 is excluded in i.

~~$B_n = B_{n-2} \left\{ \frac{n(n-1)}{40} + 33\, n(n-1) \times .0000100090 \right\}$~~

~~$\log_e B_n = \log_e B_{n-2} + \log n + \log(n-1) - 2\log 2\pi$~~

$B_n = \frac{n(n-1)}{4\pi^2} B_{n-2} \left(1 - \frac{3}{2^n-1}\right)\left(1 - \frac{8}{3^n-1}\right)\left(1 - \frac{24}{5^n-1}\right)\left(1 - \frac{48}{7^n-1}\right)$

$B_n = \frac{n(n-1)}{(n-2)(n-3)} \cdot \frac{B_{n-2} \cdot B_{n-2}}{B_{n-4}} \Pi \left\{ 1 + \frac{(p^2-1)^2}{(p^n - 1 - p^4 + \frac{p^4}{p^n})} \right\}$

$(n+\tfrac{1}{2}) \log_{10} n - 1.2324743503\, n + .7001200 + \frac{.0362}{n}$

$4 \left\{ B_2 \frac{x^2}{\lfloor 2} + B_6 \frac{x^6}{\lfloor 6} + B_{10} \frac{x^{10}}{\lfloor 10} + \&c \right\}$

$= x \cdot \frac{\frac{x^3}{\lfloor 3} - \frac{x^7}{2^4 \lfloor 7} + \frac{x^{11}}{2^4 \lfloor 11} - \&c}{\frac{x^2}{\lfloor 2} - \frac{x^6}{2^4 \lfloor 6} + \frac{x^{10}}{2^4 \lfloor 10} - \&c}$

$2 \left\{ B_0 + B_4 \frac{x^4}{\lfloor 4} + B_8 \frac{x^8}{\lfloor 8} + \&c \right\}$

$= -x \cdot \frac{\frac{x}{\lfloor 1} - \frac{x^5}{2^4 \lfloor 5} + \frac{x^9}{2^4 \lfloor 9} - \&c}{\frac{x^2}{\lfloor 2} - \frac{x^6}{2^4 \lfloor 6} + \frac{x^{10}}{2^4 \lfloor 10} - \&c}$

43C

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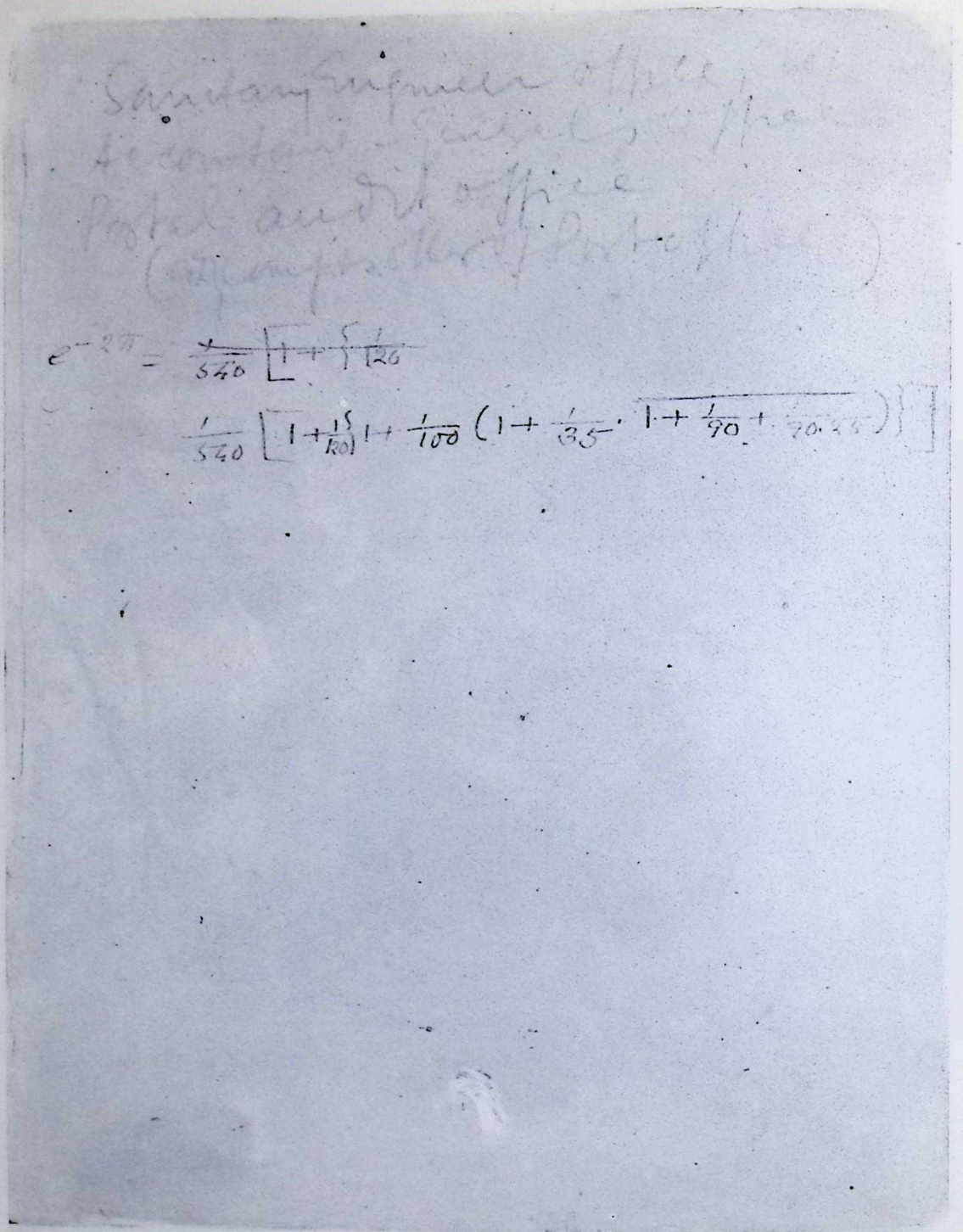

$$e^{-2\pi} = \frac{1}{540}\left[1 + \frac{5}{120}\left\{1 + \frac{1}{100}\left(1 + \frac{1}{35}\cdot\left(1 + \frac{1}{90}\cdot(\ldots)\right)\right)\right\}\right]$$

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$\sqrt{2} = 1.414213562373095048801?$

~~$\sqrt{3} = 1.7320$~~

3 — $64x^{24}$

7 — $\frac{x^{24}}{64}$

3 — $x^3 = \frac{1}{2}$

11 — $x^3 - x^2 + x = \frac{1}{2}$

19 — $x^3 + x^2 = \frac{1}{2}$

27 — $x^3 + x - \sqrt[3]{3} = \frac{1}{2}$

35 —

43 — $x^3 + x = \frac{1}{2}$

51 —

59 —

67 — $x^3 + x^2 + x = \frac{1}{2}$

75 —

7 — $x^3 = 1$

15 —

23 — $x^3 + x^2 = 1$

31 — $x^3 + x = 1$

39 —

47

55

63

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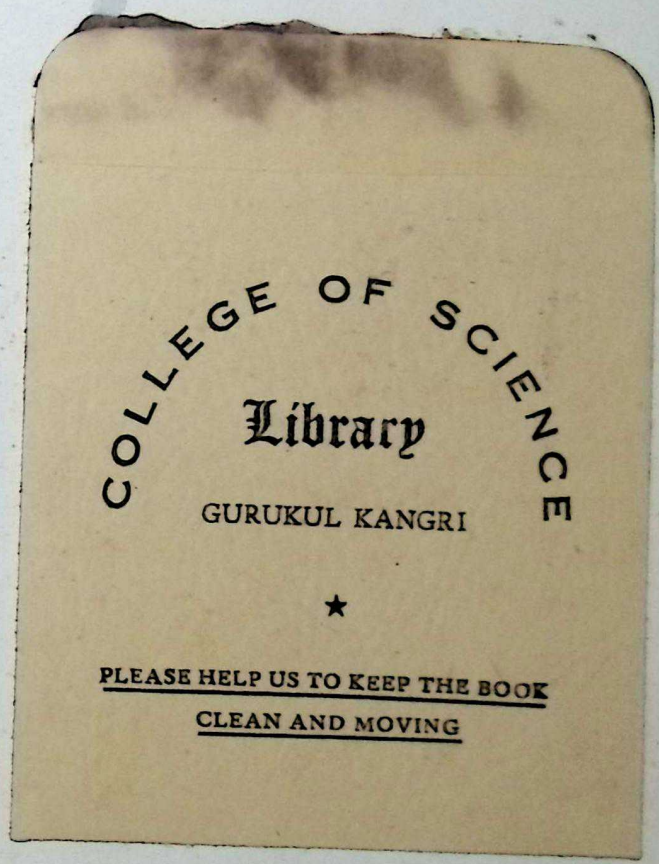

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